mathematics

MDPI

Article

A Complete Review of the General Quartic Equation with Real Coefficients and Multiple Roots

Mauricio Chávez-Pichardo [1,*], Miguel A. Martínez-Cruz [2], Alfredo Trejo-Martínez [1], Daniel Martínez-Carbajal [1] and Tanya Arenas-Resendiz [3,4]

[1] TecNM—Tecnológico de Estudios Superiores del Oriente del Estado de México, División de Estudios de Posgrado e Investigación y División de Ingeniería en Energías Renovables, La Paz C.P. 56400, Mexico; posgrado@tesoem.edu.mx (A.T.-M.); daniel.mc@otemexico.tecnm.mx (D.M.-C.)
[2] Instituto Politécnico Nacional, SEPI-ESIME Zacatenco, Unidad Profesional Adolfo López Mateos, Mexico City C.P. 07738, Mexico; mamartinezc@ipn.mx
[3] Instituto de Estudios Superiores de la Ciudad de México Rosario Castellanos IESRC, Mexico City C.P. 02440, Mexico; tanya.arenas@c3.unam.mx
[4] Centro de Ciencias de la Complejidad C3-UNAM, Mexico City C.P. 04510, Mexico
* Correspondence: mauricio.chavez@tesoem.edu.mx

Abstract: This paper presents a general analysis of all the quartic equations with real coefficients and multiple roots; this analysis revealed some unknown formulae to solve each kind of these equations and some precisions about the relation between these ones and the Resolvent Cubic; for example, it is well-known that any quartic equation has multiple roots whenever its Resolvent Cubic also has multiple roots; however, this analysis reveals that any non-biquadratic quartic equation and its Resolvent Cubic always have the same number of multiple roots; additionally, the four roots of any quartic equation with multiple roots are real whenever some specific forms of its Resolvent Cubic have three non-negative real roots. This analysis also proves that any method to solve third-degree equations is unnecessary to solve quartic equations with multiple roots, despite the existence of the Resolvent Cubic; finally, here is developed a generalized variation of the Ferrari Method and the Descartes Method, which help to avoid complex arithmetic operations during the resolution of any quartic equation with real coefficients, even though this equation has non-real roots; and a new, more simplified form of the discriminant of the quartic equations is also featured here.

Keywords: quartic equations; biquadratic equations; Resolvent Cubic; multiple roots; algebraic equations

MSC: 12D10; 26C05; 26C10

Citation: Chávez-Pichardo, M.; Martínez-Cruz, M.A.; Trejo-Martínez, A.; Martínez-Carbajal, D.; Arenas-Resendiz, T. A Complete Review of the General Quartic Equation with Real Coefficients and Multiple Roots. *Mathematics* **2022**, *10*, 2377. https://doi.org/10.3390/math10142377

Academic Editor: Dongfang Li

Received: 3 June 2022
Accepted: 30 June 2022
Published: 6 July 2022

1. Introduction

Although the resolution of third- and fourth-degree equations are usually studied in some basic college courses, the analysis exposed in this paper reveals some unknown formulae and precisions about the relation between any quartic equation with multiple roots and its corresponding Resolvent Cubic; in addition, this article is the first part of a complete review of all the fourth-degree polynomial equations, because some of the conclusions that are stated here can also be extended to the quartic equations without multiple roots.

Now, in this paper, the General Quartic Equation (GQE) will be considered as any equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$, with $a, b, c, d, e \in \mathbb{R}$ and $a \neq 0$; so, here is a complete analysis of all the cases of the GQE with multiple roots based only on the use of classical algebra tools. Hence, this analysis considers only the quartic equations that can also be expressed as only one of the following four possible forms:

- Multiplicity-4 (M4): $a(x-l)^4 = 0$ with $l \in \mathbb{R}$;
- Multiplicity-3 (M3): $a(x-l)^3(x-m) = 0$, with $l, m \in \mathbb{R}$, such that $l \neq m$;
- Double Multiplicity-2 (DM2): $a(x-l)^2(x-m)^2 = 0$, with $l, m \in \mathbb{R}$, such that $l \neq m$; or with $l, m \in \mathbb{C} - \mathbb{R}$, such that $l = \overline{m}$;
- Single Multiplicity-2 (SM2): $a(x-l)^2(x-m)(x-n) = 0$, with $l, m, n \in \mathbb{R}$, which are all different among them; or with $l \in \mathbb{R}$ and $m, n \in \mathbb{C} - \mathbb{R}$, such that $m = \overline{n}$.

For this purpose, it is also necessary to consider the Depressed Quartic Equation (DQE), which has the form $y^4 + py^2 + qy + r = 0$, with $p, q, r \in \mathbb{R}$; and this one is obtained from the GQE, after dividing each one of its members by a and applying the following change of variable:

$$x = y - \frac{b}{4a}\,; \tag{1}$$

hence, the coefficients of both equations are related as follows:

$$p = \frac{8ac - 3b^2}{8a^2}\,, \tag{2}$$

$$q = \frac{b^3 - 4abc + 8a^2d}{8a^3}\,, \tag{3}$$

$$r = \frac{16ab^2c - 64a^2bd - 3b^4 + 256a^3e}{256a^4}\,; \tag{4}$$

so, Equation (3) allows to classify the GQE in the following two complementary cases:

1. ***Biquadratic Case*:** If $q = 0$, then the corresponding DQE is reduced to $y^4 + py^2 + r = 0$, which is a biquadratic equation; thus, this equation can be easily solved by the Quadratic Formula as follows:

$$y_{1,2} = \pm\sqrt{\frac{-p + \sqrt{p^2 - 4r}}{2}} \text{ and } y_{3,4} = \pm\sqrt{\frac{-p - \sqrt{p^2 - 4r}}{2}}\,. \tag{5}$$

So, Equation (1) allows us to know the four roots of the GQE for this case as follows:

$$x_{1,2} = \pm\sqrt{\frac{-p + \sqrt{p^2 - 4r}}{2}} - \frac{b}{4a} \text{ and } x_{3,4} = \pm\sqrt{\frac{-p - \sqrt{p^2 - 4r}}{2}} - \frac{b}{4a}\,. \tag{6}$$

2. ***Ferrari Case*:** Named in allusion to the Italian renaissance mathematician Ludovico Ferrari, who was the first one to propose a method to solve the DQE, when $q \neq 0$ (non-biquadratic quartic equations) [1] (pp. 22–24); thus, Theorem 4 of this paper introduces a solution that is simultaneously a more general variation of this method and a variation of the solution given by Descartes for the DQE, as it is exposed in [1] (p. 66); after all, these methods require to solve the third-degree polynomial equation known as the Resolvent Cubic to obtain the roots of the GQE.

Ergo, the analysis featured here will begin with the simplest cases of quartic equations with multiple roots, which correspond to the Biquadratic Case: M4, DM2 and the SM2 equations whose DQE has zero as a multiple root. Then, the analysis will continue with the equations that correspond to the Ferrari Case: M3 and the SM2 equations whose DQE does not have zero as a multiple root; for these cases, it will also be necessary to expose some important properties of the Resolvent Cubic.

Finally, a brief review has also been included of the discriminant of these polynomial equations, as this quantity allows to know a priori whether these equations have multiple roots.

2. Biquadratic Equations with Multiple Roots

2.1. Multiplicity-4 Equations

Theorem 1. *(The General Solution of the M4 Equations) The GQE is an M4 equation if, and only if, it corresponds to the Biquadratic Case such that the coefficients of its DQE are* $p = q = r = 0$*, so the roots of the GQE are given by* $x_1 = x_2 = x_3 = x_4 = -\frac{b}{4a}$.

Proof. If the GQE is an M4 equation, then its four roots are $x_1 = x_2 = x_3 = x_4 = l$, for some $l \in \mathbb{R}$; therefore, $a(x-l)^4 = ax^4 - 4alx^3 + 6al^2x^2 - 4al^3x + al^4 = 0$; so, $b = -4al$, $c = 6al^2$, $d = -4al^3$ and $e = al^4$; on the one hand, the first of these relations implies $l = -b/4a$; and on the other, Equations (2)–(4) imply $p = q = r = 0$. Now, in order to prove the reciprocal, suppose that $p = q = r = 0$ in the DQE; then, Equation (6) implies that the roots of the GQE are $x_1 = x_2 = x_3 = x_4 = -\frac{b}{4a}$, thus the GQE is an M4 equation. □

Example 1. *Suppose that* $81x^4 + 756x^3 + 2646x^2 + 4116x + 2401 = 0$*, then* $a = 81$, $b = 756$, $c = 2646$, $d = 4116$ *and* $e = 2401$*; so, Equations (2)–(4) imply* $p = q = r = 0$*; ergo, Theorem 1 guarantees that all the roots of the given equation are given as follows:* $x_1 = x_2 = x_3 = x_4 = -\frac{756}{4(81)} = -\frac{7}{3}$*. In fact, this equation can also be expressed as* $81\left[x - \left(-\frac{7}{3}\right)\right]^4 = 0$.

2.2. Double Multiplicity-2 Equations

Theorem 2. *The GQE is a DM2 equation if, and only if, it corresponds to the Biquadratic Case such that* $p^2 = 4r > 0$.

Proof. If the GQE is a DM2 equation, then $a(x-l)^2(x-m)^2 = ax^4 - 2a(l+m)x^3 + a\left(l^2 + 4lm + m^2\right)x^2 - 2alm(l+m)x + al^2m^2 = 0$, with $l \neq m$; so, $b = -2a(l+m)$, $c = a\left(l^2 + 4lm + m^2\right)$, $d = -2alm(l+m)$ and $e = al^2m^2$. Therefore, Equations (2)–(4) imply the following relations:

$$p = \frac{8a\left[a\left(l^2 + 4lm + m^2\right)\right] - 3[-2a(l+m)]^2}{8a^2} = \frac{-l^2 + 2lm - m^2}{2} = -\frac{(l-m)^2}{2} \neq 0\,; \quad (7)$$

$$q = \frac{[-2a(l+m)]^3 - 4a[-2a(l+m)]\left[a\left(l^2 + 4lm + m^2\right)\right] + 8a^2[-2alm(l+m)]}{8a^3} = 0\,; \quad (8)$$

$$\begin{aligned} r &= \tfrac{16a[-2a(l+m)]^2\left[a\left(l^2+4lm+m^2\right)\right] - 64a^2[-2a(l+m)][-2alm(l+m)] - 3[-2a(l+m)]^4 + 256a^3\left(al^2m^2\right)}{256a^4} \\ &= \tfrac{l^4 - 4l^3m + 6l^2m^2 - 4lm^3 + m^4}{16} = \tfrac{(l-m)^4}{16} \neq 0\,. \end{aligned} \quad (9)$$

Hence, Equation (8) guarantees that the GQE corresponds to the Biquadratic Case; on the other hand, if $l, m \in \mathbb{R}$, then Equations (7) and (9) imply $p < 0$ and $r > 0$, respectively; but if $l, m \in \mathbb{C} - \mathbb{R}$, such that $l = \overline{m}$, then $l - m$ is the square root of a negative real number, according to the properties of complex numbers exposed in [2]; so in this case Equations (7) and (9) imply $p > 0$ and $r > 0$, respectively; thus, in any case, the following relation holds: $p^2 = \left[-\frac{(l-m)^2}{2}\right]^2 = \frac{(l-m)^4}{4} = 4\left[\frac{(l-m)^4}{16}\right] = 4r > 0$.

Now, in order to prove the reciprocal, suppose that $p^2 = 4r > 0 = q$; then, the DQE is given as follows: $y^4 + py^2 + qy + r = y^4 + py^2 + \frac{p^2}{4} = \left(y^2 + \frac{p}{2}\right)^2 = \left[y^2 - \left(\sqrt{-\frac{p}{2}}\right)^2\right]^2 = \left(y - \sqrt{-\frac{p}{2}}\right)^2\left(y + \sqrt{-\frac{p}{2}}\right)^2 = 0$, so this is a DM2 equation. Finally, Equation (1) allows us to extend this conclusion to the GQE. □

Corollary 1. *(The General Solution of the DM2 Equations) If the GQE is a DM2 equation, then:*

(i) *Its four roots are real if, and only if,* $p < 0$;

(ii) *Its four roots are non-real if, and only if,* $p > 0$;

(iii) *All its roots are given as follows:* $x_1 = x_3 = \sqrt{-\frac{p}{2}} - \frac{b}{4a}$ *and* $x_2 = x_4 = -\sqrt{-\frac{p}{2}} - \frac{b}{4a}$.

Proof. This is an immediate consequence of Theorem 2 and Equation (6). □

Example 2. *Suppose that* $4x^4 + 28x^3 + 33x^2 - 56x + 16 = 0$, *then* $a = 4$, $b = 28$, $c = 33$, $d = -56$ *and* $e = 16$; *so, Equations (2)–(4) imply* $p = -\frac{81}{8} < 0$, $q = 0$ *and* $r = \frac{6561}{256}$, *then* $p^2 = 4r = \frac{6561}{64} > 0$; *therefore, Theorem 2 implies that the given equation is a DM2 equation, whereas Corollary 1 guarantees that its four roots are real and they are obtained as follows*: $x_1 = x_3 = \sqrt{-\frac{(-81/8)}{2}} - \frac{28}{4(4)} = \frac{1}{2}$ *and* $x_2 = x_4 = -\sqrt{-\frac{(-81/8)}{2}} - \frac{28}{4(4)} = -4$; *in fact, the given equation can also be expressed as* $4\left(x - \frac{1}{2}\right)^2 [x - (-4)]^2 = 0$.

Example 3. *Suppose that* $x^4 - 12x^3 + 296x^2 - 1560x + 16900 = 0$, *then* $a = 1$, $b = -12$, $c = 296$, $d = -1560$ *and* $e = 16900$; *so, Equations (2)–(4) imply* $p = 242 > 0$, $q = 0$ *and* $r = 14641$, *then* $p^2 = 4r = 58564 > 0$; *therefore, Theorem 2 implies that the given equation is a DM2 equation, whereas Corollary 1 guarantees that its four roots are non-real and they are obtained as follows:* $x_1 = x_3 = \sqrt{-\frac{242}{2}} - \frac{(-12)}{4(1)} = 3 + 11i$ *and* $x_2 = x_4 = -\sqrt{-\frac{242}{2}} - \frac{(-12)}{4(1)} = 3 - 11i$; *in fact, the given equation can also be expressed as* $[x - (3 + 11i)]^2 [x - (3 - 11i)]^2 = 0$.

2.3. The SM2 Biquadractic Equations

Theorem 3. *If the GQE is an SM2 equation, then it corresponds to the Biquadratic Case if, and only if,* $p \neq q = r = 0$.

Proof. If the GQE is an SM2 equation, then:

$$\begin{aligned} a(x-l)^2(x-m)(x-n) = ax^4 - a(2l+m+n)x^3 \\ +a\left(l^2+2lm+2ln+mn\right)x^2 - a\left(l^2m+l^2n+2mn\right)x + al^2mn = 0\,; \end{aligned} \tag{10}$$

so, $b = -a(2l+m+n)$, $c = a\left(l^2+2lm+2ln+mn\right)$, $d = -al(lm+ln+2mn)$ and $e = al^2mn$; thus, Equations (3) and (4) imply the following relations:

$$\begin{aligned} q &= \frac{[-a(2l+m+n)]^3 - 4a[-a(2l+m+n)]\left[a\left(l^2+2lm+2ln+mn\right)\right] + 8a^2[-al(lm+ln+2mn)]}{8a^3} \\ &= \frac{2lm^2-m^3-4lmn+m^2n+2ln^2+mn^2-n^3}{8} = \left(l - \frac{m+n}{2}\right)\left(\frac{m-n}{2}\right)^2\,; \end{aligned} \tag{11}$$

$$\begin{aligned} r &= \frac{16a[-a(2l+m+n)]^2\left[a\left(l^2+2lm+2ln+mn\right)\right] - 64a^2[-a(2l+m+n)][-al(lm+ln+2mn)] - 3[-a(2l+m+n)]^4 - 256a^3\left(al^2mn\right)}{256a^4} \\ &= \frac{16l^4-32l^3m+8l^2m^2+8lm^3-3m^4-32l^3n+80l^2mn+40lm^2n+4m^3n+8l^2n^2-40lmn^2+14m^2n^2+8ln^3+4mn^3-3n^4}{256} \\ &= \frac{1}{16}\left(l - \frac{m+n}{2}\right)^2\left[\left(l - \frac{m+n}{2}\right)^2 - 4\left(\frac{m-n}{2}\right)^2\right]\,; \end{aligned} \tag{12}$$

Since $m \neq n$, Equations (11) and (12) guarantee that $q = 0$ whenever $2l = m + n$ and $r = 0$, whereas Theorem 1 guarantees $p \neq 0$.

Now, in order to prove the reciprocal, suppose that $p \neq q = r = 0$, then the DQE is reduced to the equation $y^4 + py^2 = y^2\left(y^2 + p\right) = 0$, whose roots are $y_1 = y_2 = 0$, $y_3 = \sqrt{-p} \neq 0$ and $y_4 = -\sqrt{-p} \neq 0$, so $y_3 \neq y_4$; thus, the DQE is an SM2 equation. Finally, Equation (1) allows us to extend this conclusion to the GQE. □

Corollary 2. *(The General Solution of the SM2 Biquadratic Equations) If the GQE is an SM2 equation that corresponds to the Biquadratic Case, then:*

(i) *Its four roots are real if, and only if,* $p < 0$;

(ii) *Its multiple roots are real and the other two are non-real if, and only if,* $p > 0$;
(iii) *All its roots are given as follows:* $x_1 = x_2 = -\frac{b}{4a}$, $x_3 = \sqrt{-p} - \frac{b}{4a}$ *and* $x_4 = -\sqrt{-p} - \frac{b}{4a}$.

Proof. This is an immediate consequence of Theorem 3 and Equation (6). □

Example 4. *Suppose that* $324x^4 - 2160x^3 + 5319x^2 - 5730x + 2275 = 0$, *then* $a = 324$, $b = -2160$, $c = 5319$, $d = -5730$ *and* $e = 2275$; *so, Equations (2)–(4) imply* $p = -\frac{1}{4} < 0$ *and* $q = r = 0$; *therefore, Theorem 3 guarantees that the given equation is an SM2 biquadratic equation, whereas Corollary 2 guarantees that its four roots are real, so they are obtained as follows*: $x_1 = x_2 = -\frac{(-2160)}{4(324)} = \frac{5}{3}$, $x_3 = \sqrt{-\left(-\frac{1}{4}\right)} - \frac{(-2160)}{4(324)} = \frac{13}{6}$ *and* $x_4 = -\sqrt{-\left(-\frac{1}{4}\right)} - \frac{(-2160)}{4(324)} = \frac{7}{6}$; *in fact, the given equation can also be expressed as* $324\left(x - \frac{5}{3}\right)^2\left(x - \frac{13}{6}\right)\left(x - \frac{7}{6}\right) = 0$.

Example 5. *Suppose that* $4x^4 + 8x^3 + 9x^2 + 5x + 1 = 0$, *then* $a = 4$, $b = 8$, $c = 9$, $d = 5$ *and* $e = 1$; *so, Equations (2)–(4) imply* $p = \frac{3}{4} > 0$ *and* $q = r = 0$; *therefore, Theorem 3 guarantees that the given equation is an SM2 biquadratic equation, whereas Corollary 2 guarantees that its multiple roots are real and the other two are non-real, so they are obtained as follows:* $x_1 = x_2 = -\frac{8}{4(4)} = -\frac{1}{2}$, $x_3 = \sqrt{-\frac{3}{4}} - \frac{8}{4(4)} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ *and* $x_4 = -\sqrt{-\frac{3}{4}} - \frac{8}{4(4)} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$; *in fact, the given equation can also be expressed as* $4\left[x - \left(-\frac{1}{2}\right)\right]^2\left[x - \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)\right]\left[x - \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)\right] = 0$.

The analysis of all the quartic equations with multiple roots that correspond to the Biquadratic Case finishes here, but before exposing the analysis of this kind of equations that correspond to the Ferrari Case, it will be revisited the resolution of the GQE.

3. General Solution for the Fourth-Degree Polynomial Equations

3.1. The Quartic General Formula

First of all, consider the coefficients of the DQE and the equation $s^3 + 2ps + \left(p^2 - 4r\right)s - q^2 = 0$, known as the Resolvent Cubic; then, define the following set:

$$S := \{s_1, s_2, s_3\} - \{0\} \subset \mathbb{C} - \{0\}, \tag{13}$$

where s_1, s_2 and s_3 are the three roots of the Resolvent Cubic, so S is the set of all the non-zero roots of the Resolvent Cubic. Additionally, define the parameter α_s as follows:

$$\alpha_s := \begin{cases} 0, & \text{if } S = \varnothing \\ -\frac{2q}{\sqrt{s}}, & \text{for } s \in S \neq \varnothing \end{cases}. \tag{14}$$

Proposition 1. $S = \varnothing$ *if, and only if,* $p = q = r = 0$.

Proof. Note that $S = \varnothing$ whenever $s_1 = s_2 = s_3 = 0$, so this happens if, and only if, the Resolvent Cubic has the form $s^3 = 0$; that is, $p = q = r = 0$. □

Proposition 2. *The GQE corresponds to the Biquadratic Case if, and only if,* $\alpha_s = 0$.

Proof. If $q = 0$, then there are two possibilities: $p = r = 0$ or at least one of the coefficients p and r is different from zero; so, for the first possibility, Proposition 1 and Equation (14) imply $S = \varnothing$, so $\alpha_s = 0$; meanwhile, for the second possibility, Proposition 1 guarantees the existence of $s \neq 0$, such that $s \in S \neq \varnothing$, thus Equation (14) implies $\alpha_s = -\frac{2(0)}{\sqrt{s}} = 0$. Now, in order to prove the reciprocal, suppose that $\alpha_s = 0$; thus, Equation (14) implies

two possibilities: $S = \varnothing$ or $-\frac{2q}{\sqrt{s}} = 0$ for some $s \in S \neq \varnothing$, then Proposition 1 implies $p = q = r = 0$ for the first possibility and $q = \left(-\frac{\sqrt{s}}{2}\right)(0) = 0$ for the second possibility. □

Theorem 4. *(The Quartic General Formula) The four roots of the GQE are given by the following general formula:*

$$x = \frac{\xi\sqrt{s} \pm \sqrt{\xi\alpha_s - 2p - s}}{2} - \frac{b}{4a}\,, \tag{15}$$

where $s = 0$ whenever $S = \varnothing$, otherwise $s \in S$; and $\xi := \pm 1$.

Proof. Consider the following three cases:

Case 1. Suppose that $p = q = r = 0$, then the three roots of the Resolvent Cubic are $s_1 = s_2 = s_3 = 0$ and Proposition 1 guarantees that this is the only case where $S = \varnothing$; additionally, Proposition 2 implies $\alpha_s = 0$. Therefore, Theorem 1 implies $\frac{\xi\sqrt{s}\pm\sqrt{\xi\alpha_s-2p-s}}{2} - \frac{b}{4a} = \frac{\xi\sqrt{0}\pm\sqrt{\xi(0)-2(0)-0}}{2} - \frac{b}{4a} = 0 - \frac{b}{4a} = -\frac{b}{4a} = x$.

Case 2. Suppose that $q = 0$ and at least one of the coefficients p and r of the DQE is different from zero; then, Proposition 2 implies $\alpha_s = 0$, so the Resolvent Cubic is reduced to $s\left[(s+p)^2 - 4r\right] = 0$, so its three roots are $s_1 = 0$ and $s_{2,3} = -p \pm 2\sqrt{r}$, thus at least one of the roots s_2 and s_3 is non-zero, because otherwise $p = q = r = 0$. Therefore, if $s \in S \neq \varnothing$ and considering that $\xi^2 = 1$, then:

$$\begin{aligned}\left(\frac{\xi\sqrt{s}\pm\sqrt{\xi\alpha_s-2p-s}}{2}\right)^2 &= \left[\frac{\xi\sqrt{-p\pm2\sqrt{r}}\pm\sqrt{\xi(0)-2p-\left(-p\pm2\sqrt{r}\right)}}{2}\right]^2 = \left(\frac{\xi\sqrt{-p\pm2\sqrt{r}}\pm\sqrt{-p\mp2\sqrt{r}}}{2}\right)^2 \\ &= \frac{-p\pm2\sqrt{r}\pm2\xi\left(\sqrt{-p\pm2\sqrt{r}}\right)\left(\sqrt{-p\mp2\sqrt{r}}\right)-p\mp2\sqrt{r}}{4} = \frac{-2p\pm2\xi\sqrt{\left(-p\pm2\sqrt{r}\right)\left(-p\mp2\sqrt{r}\right)}}{4} = \frac{-p\pm\xi\sqrt{p^2-4r}}{2}\,.\end{aligned} \tag{16}$$

Hence, the relation given by Equation (16) guarantees that Theorem 4 is equivalent to Equations (5) and (6) in this case.

Case 3. Suppose that $q \neq 0$ and let s and t be, such that:

$$p = t - s, \text{ with } s \neq 0. \tag{17}$$

Thus, the DQE can be rewritten as $y^4 + (t-s)y^2 + qy + r = y^4 + ty^2 - sy^2 + qy + r = 0$, so $y^4 + ty^2 = sy^2 - qy - r$, hence:

$$\left(y^2 + \frac{t}{2}\right)^2 = y^4 + ty^2 + \frac{t^2}{4} = sy^2 - qy - r + \frac{t^2}{4} = sy^2 - qy + \frac{t^2 - 4r}{4}\,. \tag{18}$$

Then, Equations (17) and (18) imply that the DQE is equivalent to the following equation:

$$\left(y^2 + \frac{s+p}{2}\right)^2 = s\left[y^2 - \frac{q}{s}y + \frac{(s+p)^2 - 4r}{4s}\right]. \tag{19}$$

In order to reduce the degree of Equation (19), it is desirable to have the following relation:

$$\left(y - \frac{q}{2s}\right)^2 = y^2 - \frac{q}{s}y + \frac{q^2}{4s^2} = y^2 - \frac{q}{s}y + \frac{(s+p)^2 - 4r}{4s}\,. \tag{20}$$

Thus, Equation (20) implies the equality $\frac{q^2}{4s^2} = \frac{(s+p)^2-4r}{4s}$, hence $s\left[(s+p)^2 - 4r\right] - q^2 = s^3 + 2ps^2 + \left(p^2 - 4r\right)s - q^2 = 0$; so, the Resolvent Cubic has finally appeared and this one guarantees $S \neq \varnothing$, because this equation is consistent with the initial suppositions that $q \neq 0$ and $s \neq 0$ for this case; ergo, Proposition 2 guarantees $\alpha_s = -\frac{2q}{\sqrt{s}} \neq 0$ for all $s \in S$.

After solving the Resolvent Cubic, Equation (19) is reduced to the following equation:

$$\left(y^2+\frac{s+p}{2}\right)^2 = s\left(y-\frac{q}{2s}\right)^2 ; \tag{21}$$

then, Equation (21) implies the following equality:

$$\left(y^2+\frac{s+p}{2}\right)^2 - \left[\sqrt{s}\left(y-\frac{q}{2s}\right)\right]^2 = \left[\left(y^2+\frac{s+p}{2}\right)-\sqrt{s}\left(y-\frac{q}{2s}\right)\right]\left[\left(y^2+\frac{s+p}{2}\right)+\sqrt{s}\left(y-\frac{q}{2s}\right)\right] = 0 ; \tag{22}$$

Therefore, if $\xi := \pm 1$, then Equation (22) is equivalent to the following quadratic equation:

$$\left(y^2+\frac{p+s}{2}\right) - \xi\sqrt{s}\left(y-\frac{q}{2s}\right) = y^2 - \xi\sqrt{s}y + \frac{1}{2}\left(p+s+\frac{\xi q}{\sqrt{s}}\right) = 0 . \tag{23}$$

Finally, if the Quadratic Formula is applied to Equation (23), then the roots of the DQE are obtained as follows:

$$y = \frac{-(-\xi\sqrt{s}) \pm \sqrt{(-\xi\sqrt{s})^2 - 4(1)\left[\frac{1}{2}\left(p+s+\frac{\xi q}{\sqrt{s}}\right)\right]}}{2(1)} = \frac{\xi\sqrt{s} \pm \sqrt{\xi\left(-\frac{2q}{\sqrt{s}}\right)-2p-s}}{2} = \frac{\xi\sqrt{s} \pm \sqrt{\xi\alpha_s - 2p - s}}{2} ; \tag{24}$$

So, Equations (1) and (24) imply that the four roots of the GQE, in this case, are also given by Equation (15). □

Since case 1 in the proof of Theorem 4 is equivalent to Theorem 1, it is not necessary to include an example for that case; however, example 6 illustrates case 2 (Biquadratic Case) and example 7 illustrates case 3 (Ferrari Case).

Example 6. *Suppose that* $2x^4-40x^3+297x^2-970x+1176=0$, *then* $a=2$, $b=-40$, $c=297$, $d=-970$ *and* $e=1176$; *so, Equations (2)–(4) imply* $p=-\frac{3}{2}$, $q=0$ *and* $r=\frac{1}{2}$; *therefore,* $2p=-3$, $p^2-4r=\frac{1}{4}$ *and* $-q^2=0$; *thus, Proposition 2 implies* $\alpha_s=0$ *and the Resolvent Cubic for this case is the equation* $s^3-3s^2+\frac{1}{4}s = s\left(s^2-3s+\frac{1}{4}\right)=0$, *whose roots are* $s_1=0$ *and* $s_{2,3}=\frac{3}{2}\pm\sqrt{2}\neq 0$; so $S=\{s_2,s_3\}\neq\varnothing$ *and, according to Theorem 4, the four roots of the given equation can be obtained with* $s_2\in S$ *as follows:* $x_{1,2} = \frac{\sqrt{3/2+\sqrt{2}}\pm\sqrt{0-2(-3/2)-(3/2+\sqrt{2})}}{2} - \frac{(-40)}{4(2)}$ *and* $x_{3,4} = \frac{-\sqrt{3/2+\sqrt{2}}\pm\sqrt{-0-2(-3/2)-(3/2+\sqrt{2})}}{2} - \frac{(-40)}{4(2)}$; *hence,* $x_1=1+5=6$, $x_{2,3}=5\pm\frac{1}{\sqrt{2}}$ *and* $x_4=-1+5=4$. *In fact, the given equation can also be expressed as* $2(x-6)\left[x-\left(5+\frac{1}{\sqrt{2}}\right)\right]\left[x-\left(5-\frac{1}{\sqrt{2}}\right)\right](x-4)=0$.

Example 7. *Suppose that* $8x^4-16x^3+12x^2-17x-14=0$, *then* $a=8$, $b=-16$, $c=12$, $d=-17$ *and* $e=-14$; *so, Equations (2)–(4) imply* $p=0$, $q=-\frac{13}{8}\neq 0$ *and* $r=-\frac{21}{8}$; *therefore,* $2p=0$, $p^2-4r=\frac{21}{2}$ *and* $-q^2=-\frac{169}{64}$; *thus, the Resolvent Cubic for this case is the equation* $s^3+\frac{21}{2}s-\frac{169}{64}=0$. *Then, the Tartaglia–Cardano Formulae, as they are applied in* [3] *(pp. 95–98), allow us to obtain the three roots of this equation as follows:* $A=\sqrt[3]{-\frac{(-169/64)}{2}+\sqrt{\left[\frac{(-169/64)}{2}\right]^2+\left[\frac{(21/2)}{3}\right]^3}}=2$, $B=\sqrt[3]{-\frac{(-169/64)}{2}-\sqrt{\left[\frac{(-169/64)}{2}\right]^2+\left[\frac{(21/2)}{3}\right]^3}}=-\frac{7}{4}$ *and* $\omega=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$; *so,* $s_1=A+B=2+\left(-\frac{7}{4}\right)=\frac{1}{4}$, $s_2=\omega A+\omega^2 B=\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(2)+\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(-\frac{7}{4}\right)=-\frac{1}{8}+\frac{15\sqrt{3}}{8}i$ *and* $s_3=\omega^2 A+\omega B=\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)(2)+\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(-\frac{7}{4}\right)=-\frac{1}{8}-\frac{15\sqrt{3}}{8}i$; *thus,* $S=\{s_1,s_2,s_3\}\neq\varnothing$ *and Proposition 2 guarantees that* $\alpha_s=-\frac{2q}{\sqrt{s}}\neq 0$ *for all* $s\in S$; *therefore, if* $s=s_1$, *then* $\alpha_s=-\frac{2(-13/8)}{\sqrt{1/4}}=\frac{13}{2}$ *and, according to Equation (15), the four roots of the given equation are obtained as follows:* $x_{1,2}=\frac{\sqrt{1/4}\pm\sqrt{13/2-2(0)-1/4}}{2}-\frac{(-16)}{4(8)}$ *and* $x_{3,4}=\frac{-\sqrt{1/4}\pm\sqrt{-13/2-2(0)-1/4}}{2}-\frac{(-16)}{4(8)}$; *hence,*

$x_1 = \frac{1/2+5/2}{2} + \frac{1}{2} = 2$, $x_2 = \frac{1/2-5/2}{2} + \frac{1}{2} = -\frac{1}{2}$ *and* $x_{3,4} = \frac{-1/2 \pm 3\sqrt{3}i/2}{2} + \frac{1}{2} = \frac{1}{4} \pm \frac{3\sqrt{3}}{4}i$. *In fact, the given equation can also be expressed as* $8(x-2)\left[x - \left(-\frac{1}{2}\right)\right]\left[x - \left(\frac{1}{4} + \frac{3\sqrt{3}}{4}i\right)\right]\left[x - \left(\frac{1}{4} - \frac{3\sqrt{3}}{4}i\right)\right] = 0$.

3.2. Some Relevant Facts about the Resolvent Cubic

Now, it will be reviewed some fundamental properties of the Resolvent Cubic that hold only for the Ferrari Case, and they are often overlooked or even ignored.

Lemma 1. *(Properties of the Resolvent Cubic for the Ferrari Case) If the GQE corresponds to the Ferrari Case, then:*

(i) The three roots of its Resolvent Cubic are different from zero;
(ii) This is the only case where $S = \{s_1, s_2, s_3\}$ *and* $\alpha_s \neq 0$ *for all* $s \in S$*;*
(iii) Its Resolvent Cubic has at least one positive real root.

Proof.

(i) This holds because $s = 0$ if, and only if, $q = 0$ in the Resolvent Cubic.
(ii) This is an immediate consequence of (i) and Equations (13) and (14).
(iii) If it is considered that the Resolvent Cubic is a polynomial equation with odd degree, then it has at least one real root, according to [3] (p. 68); in addition, if s_1, s_2 and s_3 are its three roots; then, according to [4,5], the Viète Theorem implies the following relations between the coefficients and the roots of the Resolvent Cubic: $s_1 + s_2 + s_3 = -2p$, $s_1s_2 + s_1s_3 + s_2s_3 = p^2 - 4r$ and $s_1s_2s_3 = q^2$; therefore, the last relation and (i) guarantee $q^2 = s_1s_2s_3 > 0$ for any Ferrari Case; ergo, only the following two cases are possible:

Real Case. If $s_1, s_2, s_3 \in \mathbb{R} - \{0\}$, then the law of signs guarantees that the roots of the Resolvent Cubic are all positive or at least one of them is positive, whereas the other two are negative.

Complex Case. If $s_1 \in \mathbb{R} - \{0\}$ and $s_2, s_3 \in \mathbb{C} - \mathbb{R}$, such that $s_2 = \overline{s_3}$; then $s_2s_3 = |s_2|^2 = |s_3|^2 > 0$, so according to properties of complex numbers exposed in [2], this implies $q^2 = s_1s_2s_3 = s_1|s_2|^2 = s_1|s_3|^2 > 0$, thus $s_1 > 0$. □

Remark 1. *The relevance of Lemma 1 lies with the three following important facts:*

- *It guarantees that the Ferrari Case is the only one in which any root of the Resolvent Cubic can work as s in Equation (15).*
- *The uneasy possibility of* $\sqrt{s} \notin \mathbb{R}$ *during the application of Theorem 4 is always avoidable for any Ferrari Case.*
- *In general, Lemma 1 allows us to assert that the Resolvent Cubic of any quartic equation with real coefficients has at least one non-negative real root; it does not matter if this one corresponds to the Biquadratic Case or the Ferrari Case.*

4. Non-Biquadratic Quartic Equations with Multiple Roots

4.1. Multiplicity-3 Equations

Theorem 5. *The GQE is an M3 equation if, and only if, it corresponds to the Ferrari Case and its Resolvent Cubic also has three multiple real roots, which are given as follows:* $s_1 = s_2 = s_3 = -\frac{2}{3}p > 0$.

Proof. If the GQE is an M3 equation, then its roots are $x_1 = x_2 = x_3 = l \neq x_4 = m$ for some $l, m \in \mathbb{R}$; thus:

$$a(x-l)^3(x-m) = ax^4 - a(3l+m)x^3 + 3al(l+m)x^2 - al^2(l+3m)x + al^3m = 0; \quad (25)$$

so $b = -a(3l + m)$, $c = 3al(l + m)$, $d = -al^2(l + 3m)$ and $e = al^3m$. Subsequently, define $u := \left(\frac{l-m}{2}\right)^2 > 0$; therefore, this definition, the coefficients of Equations (2)–(4) and (25) imply the following relations:

$$p = \frac{8a[3al(l+m)] - 3[-a(3l+m)]^2}{8a^2} = \frac{-3l^2 + 6lm - 3m^2}{8} = -\frac{3}{2}\left(\frac{l-m}{2}\right)^2 = -\frac{3}{2}u < 0, \tag{26}$$

$$q = \frac{[-a(3l+m)]^3 - 4a[-a(3l+m)][3al(l+m)] + 8a^2\left[-al^2(l+3m)\right]}{8a^3} = \frac{l^3 - 3l^2m + 3lm^2 - m^3}{8} = \left(\frac{l-m}{2}\right)^3 = (\pm\sqrt{u})^3 \neq 0, \tag{27}$$

$$\begin{aligned} r &= \frac{16a[-a(3l+m)]^2[3al(l+m)] - 64a^2[-a(3l+m)]\left[-al^2(l+3m)\right] - 3[-a(3l+m)]^4 + 256a^3\left(al^3m\right)}{256a^4} \\ &= \frac{-3l^4 + 12l^3m - 18l^2m^2 + 12lm^3 - 3m^4}{256} = -\frac{3}{16}\left(\frac{l-m}{2}\right)^4 = -\frac{3}{16}u^2 < 0; \end{aligned} \tag{28}$$

Hence, Equation (27) guarantees that the GQE can only correspond to the Ferrari Case; in addition, Equations (26)–(28) imply that the coefficients of the Resolvent Cubic are given as follows: $2p = 2\left(-\frac{3u}{2}\right) = -3u$, $p^2 - 4r = \left(-\frac{3u}{2}\right)^2 - 4\left(-\frac{3u^2}{16}\right) = 3u^2$ and $-q^2 = -\left[\left(\pm\sqrt{u}\right)^3\right]^2 = -u^3$; thus:

$$s^3 + 2ps^2 + \left(p^2 - 4r\right)s - q^2 = s^3 - 3us^2 + 3u^2s - u^3 = (s-u)^3 = 0. \tag{29}$$

Therefore, Equations (26) and (29) imply that the roots of the Resolvent Cubic are given as follows: $s_1 = s_2 = s_3 = u = -\frac{2p}{3} > 0$.

Now, in order to prove the reciprocal, suppose that the GQE corresponds to the Ferrari Case and that the three roots of its Resolvent Cubic are $s_1 = s_2 = s_3 = k$, for some $k \in \mathbb{R}$; then, (iii) of Lemma 1 guarantees $k > 0$, in additions, $(s-k)^3 = s^3 - 3ks^2 + 3k^2s - k^3 = s^3 + 2ps^2 + \left(p^2 - 4r\right)s - q^2 = 0$; so, $2p = -3k$ implies $k = -\frac{2p}{3}$ and $p < 0$; also, the relations $p^2 - 4r = 3k^2 = 3\left(-\frac{2p}{3}\right)^2 = \frac{4p^2}{3}$ and $q^2 = k^3 = \left(-\frac{2p}{3}\right)^3$ imply $r = -\frac{p^2}{12} < 0$ and $q = \pm\sqrt{-\frac{8p^3}{27}} = \pm\sqrt{\left(-\frac{2p}{3}\right)^3} \neq 0$, respectively; thus, the corresponding DQE can be expressed as follows:

$$y^4 + py^2 \pm \sqrt{\left(-\frac{2p}{3}\right)^3}\,y - \frac{p^2}{12} = \left(y \pm \sqrt{-\frac{3p}{2}}\right)\left(y \mp \sqrt{-\frac{p}{6}}\right)^3 = 0; \tag{30}$$

additionally, $p < 0$ implies that the expressions $\sqrt{\left(-\frac{2p}{3}\right)^3}$, $\sqrt{-\frac{3p}{2}}$ and $\sqrt{-\frac{p}{6}}$ correspond to strictly positive real numbers, so $\pm\sqrt{-\frac{3p}{2}} \neq \mp\sqrt{-\frac{p}{6}}$ in Equation (30); hence, the DQE is an M3 equation. Finally, Equation (1) allows to extend this conclusion to the GQE. □

Although Theorem 5 implies that the conditions $p < 0$, $q \neq 0$ and $r < 0$ for the coefficients of the DQE are required for the GQE to be an M3 equation, those conditions are not enough to determine certainly whether these equations are M3 equations; thus, the following corollary features a simple criterion to determine this fact with precision.

Corollary 3. *(The General Solution of the M3 Equations) The GQE is an M3 equation if, and only if, the coefficients of the corresponding DQE satisfy the conditions* $p^2 = -12r > 0$ *and* $27q^2 = -8p^3 > 0$; *whereas the roots of the GQE are given as follows*:

(i) $x_1 = x_2 = x_3 = \sqrt{-\frac{p}{6}} - \frac{b}{4a}$ and $x_4 = -\sqrt{-\frac{3p}{2}} - \frac{b}{4a}$, when $q > 0$.

(ii) $x_1 = x_2 = x_3 = -\sqrt{-\frac{p}{6}} - \frac{b}{4a}$ and $x_4 = \sqrt{-\frac{3p}{2}} - \frac{b}{4a}$, when $q < 0$.

Proof. According to Equation (30), the GQE is an M3 equation when the coefficients of the corresponding DQE are related as follows: $q = \pm\sqrt{-\frac{8p^3}{27}}$ and $r = -\frac{p^2}{12}$; hence, these relations imply these other ones: $27q^2 = -8p^3 > 0$ and $p^2 = -12r > 0$, respectively. Additionally, if $q = \sqrt{-\frac{8p^3}{27}} > 0$, then Equation (30) implies that the roots of the DQE are $y_1 = y_2 = y_3 = -\sqrt{-\frac{p}{6}}$ and $y_4 = \sqrt{-\frac{3p}{2}}$; on the other hand, if $q = -\sqrt{-\frac{8p^3}{27}} < 0$, then Equation (30) implies that the roots of the DQE are $y_1 = y_2 = y_3 = -\sqrt{-\frac{p}{6}}$ and $y_4 = \sqrt{-\frac{3p}{2}}$; finally, Equation (1) allows to obtain the roots of the GQE. □

Example 8. *Suppose that* $y^4 - 24y^2 + 64y - 48 = 0$, *then* $p = -24$, $q = 64 > 0$ *and* $r = -48$; *so*, $27q^2 = -8p^3 = 110592 > 0$ *and* $p^2 = -12r = 576 > 0$; *therefore, Corollary 3 guarantees that the given equation is an M3 equation whose roots are* $y_1 = y_2 = y_3 = \sqrt{-\frac{(-24)}{6}} = 2$ *and* $y_4 = -\sqrt{-\frac{3(-24)}{2}} = -6$. *In fact, the given equation can also be expressed as* $(y-2)^3[y-(-6)] = 0$; *additionally*, $2p = -48$, $p^2 - 4r = 2496$ *and* $-q^2 = -4096$; *then, the Resolvent Cubic for this case is the equation* $s^3 - 48s^2 + 768s - 4096 = (s-16)^3 = 0$, *whose roots are* $s_1 = s_2 = s_3 = 16$, *so this agrees with Theorem 5.*

Example 9. *Suppose that* $8x^4 - 132x^3 + 690x^2 - 1475x + 1125 = 0$, *then* $a = 8$, $b = -132$, $c = 690$, $d = -1475$ *and* $e = 1125$; *so Equations (2)–(4) imply* $p = -\frac{507}{32}$, $q = -\frac{2197}{64} < 0$ *and* $r = -\frac{85683}{4096}$; *therefore*, $27q^2 = -8p^3 = \frac{130323843}{4096} > 0$ *and* $p^2 = -12r = \frac{257049}{1024} > 0$; *then, Corollary 3 guarantees that the given equation is an M3 equation, whose roots are* $x_1 = x_2 = x_3 = -\sqrt{-\frac{(-507/32)}{6}} - \frac{(-132)}{4(8)} = \frac{5}{2}$ *and* $x_4 = \sqrt{-\frac{3(-507/32)}{2}} - \frac{(-132)}{4(8)} = 9$. *In fact, the given equation can also be expressed as* $8\left(x - \frac{5}{2}\right)^3(x-9) = 0$; *additionally*, $2p = -\frac{507}{16}$, $p^2 - 4r = \frac{85683}{256}$ *and* $-q^2 = -\frac{4826809}{4096}$; *then, the Resolvent Cubic for this case is the equation* $s^3 - \frac{507}{16}s^2 + \frac{85683}{256}s - \frac{4826809}{4096} = \left(s - \frac{169}{16}\right)^3 = 0$, *whose roots are* $s_1 = s_2 = s_3 = \frac{169}{16}$, *so this agrees with Theorem 5.*

4.2. The SM2 Non-Biquadratic Equations and the Quartic Discriminant Criteria

Theorem 6. *If the GQE is an SM2 equation, then this equation corresponds to the Ferrari Case if, and only if, its Resolvent Cubic has three real roots, such that only two of them are multiple roots; in addition, there exist the following relations between the natures of the roots of both equations:*

(i) *All the roots of the GQE are real if, and only if, all the roots of its Resolvent Cubic are positive.*
(ii) *The multiple roots of the GQE are its only real roots if, and only if, the multiple roots of its Resolvent Cubic are negative.*

Proof. If the GQE is an SM2 equation that corresponds to the Ferrari Case, then $2l \neq m + n$ in Equations (10)–(12).

Subsequently, define $v := \left(l - \frac{m+n}{2}\right)^2 \neq 0$ and $w := \left(\frac{m-n}{2}\right)^2 \neq 0$; thus, $v \neq w$ as well, because otherwise $l = m$ or $l = n$, so Equation (10) would not be an SM2 equation; likewise, note that $m + n \in \mathbb{R}$ in any case of the SM2 equations, so v is always a strictly positive real number; whereas $w > 0$ whenever $m, n \in \mathbb{R}$, and $w < 0$ whenever $m, n \in \mathbb{C} - \mathbb{R}$, such that $m = \overline{n}$, according to the properties of complex numbers exposed in [2]. Furthermore, Equations (10)–(12) imply the following relations:

$$q = \left(l - \frac{m+n}{2}\right)\left(\frac{m-n}{2}\right)^2 = \pm\sqrt{v}w \neq 0, \tag{31}$$

$$r = \frac{1}{16}\left(l - \frac{m+n}{2}\right)^2\left[\left(l - \frac{m+n}{2}\right)^2 - 4\left(\frac{m-n}{2}\right)^2\right] = \frac{v(v-4w)}{16}; \tag{32}$$

In addition, Equation (2) and the coefficients of Equation (10) imply this other relation:

$$p = \frac{8a\left[a\left(l^2+2lm+2ln+mn\right)\right] - 3[-a(2l+m+n)]^2}{8a^2} = \frac{-4l^2+4lm-3m^2+4ln+2mn-3n^2}{8} = -\left[\frac{1}{2}\left(l - \frac{m+n}{2}\right)^2 + \left(\frac{m-n}{2}\right)^2\right] = -\left(\frac{v}{2}+w\right); \tag{33}$$

Thus, the coefficients of the Resolvent Cubic are $2p = 2\left[-\left(\frac{v}{2}+w\right)\right] = -(v+2w)$, $p^2 - 4r = \left[-\left(\frac{v}{2}+w\right)\right]^2 - 4\left[\frac{v(v-4w)}{16}\right] = (2v+w)w$ and $-q^2 = -\left(\pm\sqrt{v}w\right)^2 = -vw^2$; then:

$$s^3 + 2ps^2 + \left(p^2 - 4r\right)s - q^2 = s^3 - (v+2w)s^2 + [(2v+w)w]s - vw^2 = (s-v)(s-w)^2 = 0; \tag{34}$$

So, the roots of this equation are $s_1 = v > 0$ and $s_2 = s_3 = w \neq 0$; also, note that s_1 is the root mentioned in (iii) of Lemma 1, whereas $s_2 = s_3 > 0$ if, and only if, $m, n \in \mathbb{R}$, so in that case all the roots of the GQE are real; on the other hand, $s_2 = s_3 < 0$ if, and only if, m, $n \in \mathbb{C} - \mathbb{R}$, so in that case, the multiple roots of the GQE are its only real roots.

Now, in order to prove the reciprocal, suppose that there exist $k_1, k_2 \in \mathbb{R} - \{0\}$, such that $k_1 \neq k_2$, and that the roots of the Resolvent Cubic are $s_1 = k_1 > 0$ and $s_2 = s_3 = k_2$; then, $(s-k_1)(s-k_2)^2 = s^3 - (k_1+2k_2)s^2 + (2k_1+k_2)k_2 s - k_1k_2^2 = s^3 + 2ps^2 + \left(p^2-4r\right)s - q^2 = 0$; therefore, $2p = -(k_1+2k_2)$, $p^2 - 4r = (2k_1+k_2)k_2$ and $q^2 = k_1k_2^2 > 0$; thus, $p = -\left(\frac{k_1}{2}+k_2\right)$, $q = \pm\sqrt{k_1}k_2 \neq 0$ and $r = \frac{k_1}{4}\left(\frac{k_1}{4} - k_2\right)$; so the GQE corresponds to the Ferrari Case, and the respective DQE is given as follows:

$$y^4 - \left(\frac{k_1}{2}+k_2\right)y^2 \pm \sqrt{k_1}k_2 y + \frac{k_1}{4}\left(\frac{k_1-4k_2}{4}\right) = \left(y \mp \frac{\sqrt{k_1}}{2}\right)^2\left(y^2 \pm \sqrt{k_1}y + \frac{k_1-4k_2}{4}\right) = 0. \tag{35}$$

Hence, the four roots of the DQE are $y_1 = y_2 = \frac{\sqrt{k_1}}{2}$, $y_3 = -\frac{\sqrt{k_1}}{2} + \sqrt{k_2}$ and $y_4 = -\frac{\sqrt{k_1}}{2} - \sqrt{k_2}$, whenever $q = \sqrt{k_1}k_2$; whereas they are $y_1 = y_2 = -\frac{\sqrt{k_1}}{2}$, $y_3 = \frac{\sqrt{k_1}}{2} + \sqrt{k_2}$ and $y_4 = \frac{\sqrt{k_1}}{2} - \sqrt{k_2}$, whenever $q = -\sqrt{k_1}k_2$; so, it is clear that in any case $y_1, y_2 \in \mathbb{R}$; however, $y_3, y_4 \in \mathbb{R}$ if, and only if, $k_2 > 0$; and $y_3, y_4 \in \mathbb{C} - \mathbb{R}$ if, and only if, $k_2 < 0$. Finally, Equation (1) allows extending these conclusions to the roots of the GQE. □

Before continuing with the analysis of the SM2 non-biquadratic equations, it will be exposed some important facts involving the discriminant of the quartic equations.

Lemma 2. *(Discriminant Criterion for the Quartic Equations with Multiple Roots) The GQE has multiple roots if, and only if, the coefficients of the corresponding DQE are related as follows:* $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right) + \frac{27}{2}q^2\right]^2$.

Proof. *Biquadratic Case*. Theorems 1, 2 and 3 guarantee that the GQE has multiple roots if, and only if, $\left(p^2-4r\right)r = 0 = q$; so, $0 = 108\left(p^2-4r\right)^2 r = 108p^4r - 864p^2r^2 + 1728r^3 = \left(p^2+12r\right)^3 - \left[p\left(p^2-36r\right)\right]^2$, therefore $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+0\right]^2 = \left[p\left(p^2-36r\right) + \frac{27}{2}q^2\right]^2$.

Ferrari Case. If the change of variable $s = z - \frac{2p}{3}$ is applied to the Resolvent Cubic, then:

$$s^3 + 2ps^2 + \left(p^2-4r\right)s - q^2 = \left(z - \frac{2p}{3}\right)^3 + 2p\left(z - \frac{2p}{3}\right)^2 + \left(p^2-4r\right)\left(z - \frac{2p}{3}\right) - q^2 = z^3 + p^*z + q^* = 0, \\ \text{where } p^* := -\frac{p^2+12r}{3} \text{ and } q^* := -\frac{2p^3-72pr+27q^2}{27}; \tag{36}$$

Therefore, according to [6], the discriminant of the Resolvent Cubic is given as follows:

$$\Delta = -4{p^*}^3 - 27{q^*}^2 = -4\left(-\frac{p^2+12r}{3}\right)^3 - 27\left(-\frac{2p^3-72pr+27q^2}{27}\right)^2 = \frac{4\left(p^2+12r\right)^3 - \left[2p\left(p^2-36r\right)+27q^2\right]^2}{27}. \quad (37)$$

Then, Theorems 5 and 6 reassert the previously known fact that the GQE has multiple roots if, and only if, the Resolvent Cubic also has multiple roots, as stated in [7]; and, according to [1] (p.103), this happens if, and only if, $\Delta = 0$; thus, in that case, Equation (37) implies the following relation: $\left(p^2+12r\right)^3 = \frac{\left[2p\left(p^2-36r\right)+27q^2\right]^2}{4} = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2$. □

It is included a justification for the name given to Lemma 2 in Appendix A.

Corollary 4. *If the GQE corresponds to a Biquadratic Case with multiple roots, then:*

(i) *It is an M4 equation if, and only if,* $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 = 0$;

(ii) *It is a DM2 equation if, and only if,* $(2p)^6 = \left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 = 2^{12}r^3 > 0$;

(iii) *It is an SM2 equation if, and only if,* $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 = p^6 > 0$.

Proof. All these relations are immediate consequences of Lemma 2 and Theorems 1, 2 and 3. □

Theorem 7. *If the GQE corresponds to a Ferrari Case with multiple roots, then:*

(i) *It is an M3 equation if, and only if,* $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 = 0$;

(ii) *It is an SM2 equation if, and only if,* $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 > 0$.

Proof.

(i) According to Theorem 5, the GQE is an M3 equation if, and only if, the roots of its Resolvent Cubic are $s_1 = s_2 = s_3 = -\frac{2p}{3}$, and it occurs whenever the roots of Equation (36) are $z_1 = z_2 = z_3 = 0$ and $\Delta = p^* = q^* = 0$; so this happens if, and only if, $\left(-3p^*\right)^3 = \left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 = \left(\frac{27}{2}q^*\right)^2 = 0$.

(ii) Lemma 2 and (i) guarantee that the GQE is an SM2 equation that corresponds to the Ferrari Case if, and only if, $p^* \neq 0 \neq q^*$ in Equation (36) and $\Delta = 0$ in Equation (37); so this happens if, and only if, all the three following relations hold: $-3p^* = p^2+12r \neq 0$, $-\frac{27}{2}q^* = p\left(p^2-36r\right)+\frac{27}{2}q^2 \neq 0$ and $\left(p^2+12r\right)^3 = \left[p\left(p^2-36r\right)+\frac{27}{2}q^2\right]^2 > 0$. □

Remark 2. *Note that the conditions established in Corollary 3 to identify M3 equations are equivalent to (i) of Theorem 7.*

Theorem 8. *If the GQE is an SM2 equation that corresponds to the Ferrari Case, then the three roots of its Resolvent Cubic are given as follows:*

(i) *If* $r > 0$, *then* $s_1 = \frac{2\left(-p+\sqrt{p^2+12r}\right)}{3} > 0$ *and* $s_2 = s_3 = \frac{-2p-\sqrt{p^2+12r}}{3} \neq 0$, *for any* $p \in \mathbb{R}$;

(ii) *If* $r = 0$, *then* $p < 0$, $s_1 = -\frac{4p}{3} > 0$ *and* $s_2 = s_3 = -\frac{p}{3} > 0$;

(iii) *If* $r < 0$, *then* $p < 0$, $s_1 = \frac{2\left(-p\pm\sqrt{p^2+12r}\right)}{3} > 0$ *and* $s_2 = s_3 = \frac{-2p\mp\sqrt{p^2+12r}}{3} > 0$, *such that* $s_1 s_2^2 = s_1 s_3^2 = q^2$.

Proof. If the GQE is an SM2 equation that corresponds to the Ferrari Case, then its Resolvent Cubic is given by Equation (34), whose roots are $s_1 = v > 0$ and $s_2 = s_3 = w \neq 0$; additionally, Equation (34) also implies the following non-linear system of equations:

$$v + 2w = -2p, \tag{38}$$

$$(2v + w)w = p^2 - 4r; \tag{39}$$

whose solutions are $v = \frac{2\left(-p \pm \sqrt{p^2+12r}\right)}{3}$ and $w = \frac{-2p \mp \sqrt{p^2+12r}}{3}$. Subsequently, Corollary 4 guarantees $p^2 + 12r > 0$, so $\sqrt{p^2 + 12r} \in \mathbb{R}$; thus, there are three possibilities:

(i) Suppose that $r > 0$, thus $\sqrt{p^2 + 12r} > |p| \geq 0$, then $-\sqrt{p^2 + 12r} < p < \sqrt{p^2 + 12r}$, so this implies $-p - \sqrt{p^2 + 12r} < 0 < -p + \sqrt{p^2 + 12r}$; therefore, the roots of the Resolvent Cubic for this possibility can only be given as follows: $s_1 = v = \frac{2\left(-p+\sqrt{p^2+12r}\right)}{3} > 0$ and $s_2 = s_3 = w = \frac{-2p-\sqrt{p^2+12r}}{3} \neq 0$;

(ii) Suppose that $r = 0$, then $\sqrt{p^2 + 12r} = |p|$; ergo, if $p \geq 0$, then $v = 0$ or $v = -4p/3 \leq 0$, but both relations contradict the inequality $v > 0$; thus $p < 0$, uniquely. So, $|p| = -p$ and the roots of the Resolvent Cubic for this possibility are given as follows: $s_1 = v = \frac{2\left(-p+\sqrt{p^2+12r}\right)}{3} = \frac{2(-p-p)}{3} = -\frac{4p}{3} > 0$ and $s_2 = s_3 = w = \frac{-2p-\sqrt{p^2+12r}}{3} = \frac{-2p-(-p)}{3} = -\frac{p}{3} > 0$;

(iii) Suppose that $r < 0$, thus $|p| > \sqrt{p^2 + 12r} > 0$ and $p \neq 0$; whereas $-p < -\sqrt{p^2 + 12r} < 0 < \sqrt{p^2 + 12r} < p$, whether $p > 0$; and $p < -\sqrt{p^2 + 12r} < 0 < \sqrt{p^2 + 12r} < -p$, whether $p < 0$; so, $-2p < -p - \sqrt{p^2 + 12r} < -p < -p + \sqrt{p^2 + 12r} < 0$, whether $p > 0$; and $0 < -p - \sqrt{p^2 + 12r} < -p < -p + \sqrt{p^2 + 12r} < -2p$, whether $p < 0$; hence, $v > 0$ guarantees $p < 0$, uniquely; therefore $0 < -p < -2p - \sqrt{p^2 + 12r} < -2p < -2p + \sqrt{p^2 + 12r} < -3p$, so this guarantees $w > 0$. Thus, the roots of the Resolvent Cubic are given as follows: $s_1 = v = \frac{2\left(-p\pm\sqrt{p^2+12r}\right)}{3} > 0$ and $s_2 = s_3 = w = \frac{-2p\mp\sqrt{p^2+12r}}{3} > 0$; finally, in order to decide which signs before the square roots are the suitable ones for this case, it must be considered that Equation (34) also implies the relation $-q^2 = -vw^2$; that is, $s_1 s_2^2 = s_1 s_3^2 = q^2$. □

Corollary 5. *If the GQE is an SM2 equation that corresponds to the Ferrari Case, and $r > 0$ in the corresponding DQE; then:*

(i) The three roots of its Resolvent Cubic are related as follows: $s_1 > s_2 = s_3$;

(ii) If $p \geq 0$, then $s_2 = s_3 < 0$;

(iii) If $p < 0$, then $s_2 = s_3 > 0$ if, and only if, $p^2 > 4r$.

Proof.

(i) If $r > 0$, then (i) of Theorem 8 implies $s_1 - s_2 = s_1 - s_3 = \sqrt{p^2 + 12r} > 0$, thus $s_1 > s_2 = s_3$;

(ii) If $p \geq 0$, then (i) of Theorem 8 implies $s_2 = s_3 < 0$;

(iii) Note that the relation $s_1 > s_2 = s_3$ is equivalent to the inequality $v > w$ in Equation (34), hence $v + w > 2w$; also, if $p < 0$, then Equation (38) implies $2v + w > v + 2w = -2p > 0$; so, if $2v + w > 0$, then Equation (39) guarantees that $s_2 = s_3 = w > 0$ whenever $p^2 - 4r > 0$. □

Remark 3. *Note that if the GQE is an SM2 equation that corresponds to the Ferrari Case, then Theorems 6 and 8 and Corollary 5 guarantee that its four roots are real when $r \leq 0$ or when $p < 0 < r$ with $p^2 > 4r$; on the other hand, its multiple roots are its only two real roots when $p < 0 < p^2 \leq 4r$ or when $r > 0$ with $p \geq 0$.*

Theorem 9. *If the GQE is an SM2 equation that corresponds to the Ferrari Case, then its four roots are given as follows:*

(i) $x_1 = x_2 = \frac{\sqrt{s_1}}{2} - \frac{b}{4a}$ *and* $x_{3,4} = -\frac{\sqrt{s_1}}{2} \pm \sqrt{s_2} - \frac{b}{4a}$ *, when q and* s_2 *have the same signs.*

(ii) $x_1 = x_2 = -\frac{\sqrt{s_1}}{2} - \frac{b}{4a}$ *and* $x_{3,4} = \frac{\sqrt{s_1}}{2} \pm \sqrt{s_2} - \frac{b}{4a}$ *, when q and* s_2 *have opposite signs.*

Proof. If $k_1 = s_1$ and $k_2 = s_2 = s_3$ in Equation (35), then there are four possible cases to obtain all the roots of the DQE:

Case 1. If $q = \sqrt{s_1}s_2 > 0$, then $s_2 = s_3 > 0$; so, Equation (35) implies that the roots of the DQE are $y_1 = y_2 = \frac{\sqrt{s_1}}{2}$ and $y_{3,4} = -\frac{\sqrt{s_1}}{2} \pm \sqrt{s_2}$;

Case 2. If $q = \sqrt{s_1}s_2 < 0$, then $s_2 = s_3 < 0$; so, Equation (35) also implies that the roots of the DQE are obtained as in case 1;

Case 3. If $q = -\sqrt{s_1}s_2 > 0$, then $s_2 = s_3 < 0$; so, Equation (35) implies that the roots of the DQE are $y_1 = y_2 = -\frac{\sqrt{s_1}}{2}$ and $y_{3,4} = \frac{\sqrt{s_1}}{2} \pm \sqrt{s_2}$;

Case 4. If $q = -\sqrt{s_1}s_2 < 0$, then $s_2 = s_3 > 0$; so, Equation (35) now implies that the roots of the DQE are obtained as in case 3 as well.

Finally, Equation (1) allows obtaining all the roots of the GQE after having obtained all the roots of the DQE. □

Example 10. *Suppose that* $4x^4 - 4\sqrt{2}x + 3 = 0$*, thus* $a = 4$*,* $b = c = 0$*,* $d = -4\sqrt{2}$ *and* $e = 3$*; so,* $p = 0$*,* $q = -\sqrt{2} < 0$ *and* $r = \frac{3}{4} > 0$*; therefore,* $2p = 0$*,* $p^2 - 4r = -3 < 0$*,* $-q^2 = -2$ *and* $\left(p^2 + 12r\right)^3 = \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2 = 3^6 > 0$*, then* $\sqrt{p^2 + 12r} = 3$*; hence, Corollary 4 guarantees that the given equation is an SM2 equation and, according to Remark 3, this equation has only two real roots. Additionally, (ii) of Corollary 5 implies* $s_2 = s_3 < 0$*, and (i) of Theorem 8 implies that the roots of the Resolvent Cubic are given as follows:* $s_1 = \frac{2(-0+3)}{3} = 2 > 0$ *and* $s_2 = s_3 = \frac{-2(0)-3}{3} = -1 < 0$*; in fact, the Resolvent Cubic for this case is the equation* $s^3 - 3s - 2 = (s-2)[s-(-1)]^2 = 0$*. On the other hand, q and* s_2 *are negative; thus, (ii) of Theorem 9 implies that the roots of the given equation are obtained as follows:* $x_1 = x_2 = \frac{\sqrt{2}}{2} - \frac{0}{4(4)} = \frac{1}{\sqrt{2}}$ *and* $x_{3,4} = -\frac{\sqrt{2}}{2} \pm \sqrt{-1} - \frac{0}{4(4)} = -\frac{1}{\sqrt{2}} \pm i$*; finally, note that the given equation can also be expressed as* $4\left(x - \frac{1}{\sqrt{2}}\right)^2 \left[x - \left(-\frac{1}{\sqrt{2}} + i\right)\right] \left[x - \left(-\frac{1}{\sqrt{2}} - i\right)\right] = 0$*.*

Example 11. *Suppose that* $x^4 + 14x^3 + 68x^2 + 130x + 75 = 0$*, thus* $a = 1$*,* $b = 14$*,* $c = 68$*,* $d = 130$ *and* $e = 75$*; so, Equations (2)–(4) imply* $p = -\frac{11}{2} < 0$*,* $q = -3 < 0$ *and* $r = \frac{45}{16} > 0$*; therefore,* $2p = -11$*,* $p^2 - 4r = 19 > 0$*,* $-q^2 = -9$ *and* $\left(p^2 + 12r\right)^3 = \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2 = 8^6 > 0$*, then* $\sqrt{p^2 + 12r} = 8$*; hence, Corollary 4 guarantees that the given equation is an SM2 equation and* $p < 0 < r$ *with* $p^2 = 121/4 > 45/4 = 4r$*; so, according to Remark 3, the four roots of this equation are real. Additionally, (iii) of Corollary 5 implies* $s_2 = s_3 > 0$ *and (i) of Theorem 8 implies that the roots of the Resolvent Cubic are given as follows:* $s_1 = \frac{2[-(-11/2)+8]}{3} = 9 > 0$ *and* $s_2 = s_3 = \frac{-2(-11/2)-8}{3} = 1 > 0$*; in fact, the Resolvent Cubic for this case is the equation* $s^3 - 11s^2 + 19s - 9 = (s-9)(s-1)^2 = 0$*. On the other hand, q and* s_2 *have opposite signs; thus, (ii) of Theorem 9 implies that the roots of the given equation are obtained as follows:* $x_1 = x_2 = -\frac{\sqrt{9}}{2} - \frac{14}{4(1)} = -5$ *,* $x_3 = \frac{\sqrt{9}}{2} + \sqrt{1} - \frac{14}{4(1)} = -1$ *and* $x_4 = \frac{\sqrt{9}}{2} - \sqrt{1} - \frac{14}{4(1)} = -3$*; finally, note that the given equation can also be expressed as* $[x-(-5)]^2[x-(-1)][x-(-3)] = 0$*.*

Example 12. *Suppose that* $343x^4 - 196x^3 - 294x^2 + 220x - 25 = 0$*, thus* $a = 343$*,* $b = -196$*,* $c = -294$*,* $d = 220$ *and* $e = -25$*; so, Equations (2)–(4) imply* $p = -\frac{48}{49} < 0$*,* $q = \frac{128}{343} > 0$ *and* $r = 0$*; therefore,* $2p = -\frac{96}{49}$*,* $p^2 - 4r = \frac{2304}{2401}$*,* $-q^2 = -\frac{16384}{117649}$ *and* $\left(p^2 + 12r\right)^3 = \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2 = \frac{48^6}{7^{12}} > 0$*; hence, Corollary 4 guarantees that the given equation is an SM2 equation and, according to Remark 3, its four roots are real. Additionally, (ii) of Theorem 8 implies that the roots of the Resolvent Cubic are given as follows:* $s_1 = -\frac{4(-48/49)}{3} = \frac{64}{49} >$

0 *and* $s_2 = s_3 = -\frac{(-48/49)}{3} = \frac{16}{49} > 0$; *in fact, the Resolvent Cubic for this case is the equation* $s^3 - \frac{96}{49}s^2 + \frac{2304}{2401}s - \frac{16384}{117649} = \left(s - \frac{64}{49}\right)\left(s - \frac{16}{49}\right)^2 = 0$. *On the other hand, q and* s_2 *are positive, hence (i) of Theorem 9 implies that the roots of the given equation are obtained as follows:* $x_1 = x_2 = \frac{\sqrt{64/49}}{2} - \frac{(-196)}{4(343)} = \frac{5}{7}$, $x_3 = -\frac{\sqrt{64/49}}{2} + \sqrt{\frac{16}{49} - \frac{(-196)}{4(343)}} = \frac{1}{7}$ *and* $x_4 = -\frac{\sqrt{64/49}}{2} - \sqrt{\frac{16}{49} - \frac{(-196)}{4(343)}} = -1$; *finally, note that the given equation can also be expressed as* $343\left(x - \frac{5}{7}\right)^2\left(x - \frac{1}{7}\right)[x - (-1)] = 0$.

Example 13. *Suppose that* $9x^4 - 15x^3 - 101x^2 + 71x - 12 = 0$, *thus* $a = 9$, $b = -15$, $c = -101$, $d = 71$ *and* $e = -12$; *so, Equations (2)–(4) imply* $p = -\frac{883}{72} < 0$, $q = -\frac{49}{24} < 0$ *and* $r = -\frac{1763}{20736} < 0$; *therefore,* $2p = -\frac{883}{72}$, $p^2 - 4r = \frac{21707}{144}$, $-q^2 = -\frac{2401}{576}$ *and* $\left(p^2 + 12r\right)^3 = \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2 = \frac{110^6}{3^{12}} > 0$, *then* $\sqrt{p^2 + 12r} = \frac{110}{9}$; *hence, Corollary 4 guarantees that the given equation is an SM2 equation and, according to Remark 3, its four roots are real. Additionally, (iii) of Theorem 8 implies that the roots of the Resolvent Cubic are given as follows:* $s_1 = \frac{2[-(-883/72) \pm 110/9]}{3} = \frac{883 \pm 880}{108} > 0$ *and* $s_2 = s_3 = \frac{-2(-883/72) \mp 110/9}{3} = \frac{883 \mp 440}{108} > 0$; *so, there are two possible couples of values for the roots of the Resolvent Cubic:* $s_1 = \frac{1763}{108}$ *with* $s_2 = s_3 = \frac{443}{108}$ *or* $s_1 = \frac{1}{36}$ *with* $s_2 = s_3 = \frac{49}{4}$; *in fact,* $\left(\frac{1763}{108}\right)\left(\frac{443}{108}\right)^2 = \frac{345986987}{1259712} \neq q^2$ *and* $\left(\frac{1}{36}\right)\left(\frac{49}{4}\right)^2 = \frac{2401}{576} = q^2$; *hence, the Resolvent Cubic for this case is the equation* $s^3 - \frac{883}{72}s^2 + \frac{21707}{144}s - \frac{2401}{576} = \left(s - \frac{1}{36}\right)\left(s - \frac{49}{4}\right)^2 = 0$. *On the other hand, q and* s_2 *have opposite signs, then (ii) of Theorem 9 guarantees that the roots of the given equation are obtained as follows*: $x_1 = x_2 = -\frac{\sqrt{1/36}}{2} - \frac{(-15)}{4(9)} = \frac{1}{3}$, $x_3 = \frac{\sqrt{1/36}}{2} + \sqrt{\frac{49}{4} - \frac{(-15)}{4(9)}} = 4$ *and* $x_4 = \frac{\sqrt{1/36}}{2} - \sqrt{\frac{49}{4} - \frac{(-15)}{4(9)}} = -3$; *finally, note that the given equation can also be expressed as* $9\left(x - \frac{1}{3}\right)^2(x - 4)[x - (-3)] = 0$.

4.3. The General Solutions of the SM2 Equations

Although Theorems 8 and 9 are useful to solve any SM2 equation that corresponds to the Ferrari Case, now it will be featured a more general and efficient solution for this kind of quartic equation that also works for the Biquadratic Case.

Theorem 10. *If the GQE is an SM2 equation, then the three roots of its Resolvent Cubic are given by the following general formulae:*

$$s_1 = \frac{9q^2 - 32pr}{p^2 + 12r} \geq 0, \tag{40}$$

$$s_2 = s_3 = -\frac{2p\left(p^2 - 4r\right) + 9q^2}{2\left(p^2 + 12r\right)} \neq 0. \tag{41}$$

Proof. *Biquadratic Case.* According to Theorem 3, $p \neq q = r = 0$; so, $2p \neq 0$, $p^2 - 4r = p^2 \neq 0$ and $-q^2 = 0$; therefore, the Resolvent Cubic for this case is reduced to $s^3 + 2ps^2 + p^2 s = 0$, whose roots are $s_1 = 0$ and $s_2 = s_3 = -p \neq 0$. Finally, Theorem 3 also implies $\frac{9q^2 - 32pr}{p^2 + 12r} = \frac{9(0)^2 - 32p(0)}{p^2 + 12(0)} = 0 = s_1$ and $-\frac{2p\left(p^2 - 4r\right) + 9q^2}{2\left(p^2 + 12r\right)} = -\frac{2p\left[p^2 - 4(0)\right] + 9(0)^2}{2\left[p^2 + 12(0)\right]} = -p = s_2 = s_3$.

Ferrari Case. First of all, consider that Equation (34) is the Resolvent Cubic for this case, so this equation implies the following relation:

$$q^2 = vw^2, \text{ where } s_1 = v > 0 \text{ and } s_2 = s_3 = w \neq 0. \tag{42}$$

Then, Equations (38) and (42) imply $v + 2w = q^2/w^2 + 2w = -2p$, thus:

$$2w^3 + 2pw^2 + q^2 = 0; \tag{43}$$

likewise, Equations (39) and (42) imply $(2v + w)w = 2q^2/w + w^2 = p^2 - 4r$, thus:

$$w^3 - \left(p^2 - 4r\right)w + 2q^2 = 0; \tag{44}$$

moreover, Equations (43) and (44) imply the relation $w^3 = -\left(2pw^2 + q^2\right)/2 = \left(p^2 - 4r\right)w - 2q^2$, so $2pw^2 + q^2 = -2\left(p^2 - 4r\right)w + 4q^2$; therefore:

$$2pw^2 + 2\left(p^2 - 4r\right)w - 3q^2 = 0. \tag{45}$$

On the other hand, Equations (38) and (39) imply $(2v + w)w = [2(-2w - 2p) + w]w = -3w^2 - 4pw = p^2 - 4r$, thus:

$$w^2 = -\frac{4pw + p^2 - 4r}{3}\,; \tag{46}$$

So, Equations (45) and (46) imply $2pw^2 + 2\left(p^2 - 4r\right)w - 3q^2 = 2p\left(\frac{4pw+p^2-4r}{3}\right) + 2\left(p^2 - 4r\right)w - 3q^2 = -\frac{2\left(p^2+12r\right)w+2p\left(p^2-4r\right)+9q^2}{3} = 0$, hence $s_2 = s_3 = w = -\frac{2p\left(p^2-4r\right)+9q^2}{2\left(p^2+12r\right)}$; furthermore, Equation (38) implies $s_1 = v = -2w - 2p = -2\left[-\frac{2p\left(p^2-4r\right)+9q^2}{2\left(p^2+12r\right)}\right] - 2p = \frac{9q^2-32pr}{p^2+12r}$. Finally, Corollary 3 guarantees $9q^2 - 32pr = 2p\left(p^2 - 4r\right) + 9q^2 = p^2 + 12r = 0$ whenever the GQE is an M3 equation, so Equations (40) and (41) never go undefined for the SM2 non-biquadratic equations, and it also reasserts $s_1 \neq 0 \neq s_2 = s_3$ for this case. □

Remark 4. *Note that in Theorem 10,* $s_1 = 0$ *whenever the GQE is an SM2 biquadratic equation, whereas* $s_1 > 0$ *whenever the GQE is an SM2 non-biquadratic equation.*

Corollary 6. *(The General Solution of the SM2 Equations) If the GQE is an SM2 equation, then its four roots can be obtained by the following general formula*:

$$x = \frac{1}{2}\left[\xi\sqrt{-\frac{2p(p^2-4r)+9q^2}{2(p^2+12r)}} \pm \sqrt{-2\xi q\sqrt{-\frac{2(p^2+12r)}{2p(p^2-4r)+9q^2}} - \frac{2p(p^2+14r)-9q^2}{2(p^2+12r)}}\right] - \frac{b}{4a}\,; \tag{47}$$

where $\xi = \pm 1$.

Proof. Equation (47) is a consequence of Theorem 4 and Equation (41), in addition, Theorem 10 guarantees $p^2 + 12r \neq 0 \neq 2p\left(p^2 - 4r\right) + 9q^2$, so Equation (47) never goes undefined. □

Although Corollary 6 gives a general solution for all the SM2 equations, it involves complex arithmetic operations whenever $s_2 = s_3 < 0$ in Equation (41), and these operations are often somewhat cumbersome, especially in the Ferrari Case; so, the following corollary introduces a less general but more efficient formula that works only for the SM2 non-biquadratic equations.

Corollary 7. *(The General Solution of the SM2 Non-Biquadratic Equations) If the GQE is an SM2 equation that corresponds to the Ferrari Case, then its four roots are given by the following general formula:*

$$x = \frac{1}{2}\left[\xi\sqrt{\frac{9q^2-32pr}{p^2+12r}} \pm \sqrt{-\left(2\xi q\sqrt{\frac{p^2+12r}{9q^2-32pr}} + \frac{2p(p^2-4r)+9q^2}{p^2+12r}\right)}\right] - \frac{b}{4a}, \quad (48)$$

where $\xi = \pm 1$; and this formula never requires the application of complex arithmetic operations.

Proof. Equation (48) is a consequence of Theorem 4 and Equation (40), in addition, Equation (40) and Remark 4 guarantee $\left(\frac{9q^2-32pr}{p^2+12r}\right)^{\pm 1/2} \in \mathbb{R} - \{0\}$ for any SM2 non-biquadratic equation; hence, Equation (48) never goes undefined and it also never requires the application of complex arithmetic operations; on the other hand, Theorem 3 implies $9q^2 - 32pr = 0$ for all the SM2 biquadratic equations, so Equation (48) is undefined for that kind of SM2 equations. □

According to Theorem 10, Equation (48) can be rewritten as $x = \frac{1}{2}\left[\xi\sqrt{s_1} \pm \sqrt{2\left(s_2 - \frac{\xi q}{\sqrt{s_1}}\right)}\right] - \frac{b}{4a}$; moreover, the efficacy of this theorem and Corollaries 6 and 7 can be verified applying them to Examples 4, 5 and 10–13. Nevertheless, although these results can be more effective than Theorems 8 and 9 to solve any SM2 non-biquadratic equation, Theorem 10 and Corollaries 6 and 7 would not be useful to establish Remark 3.

Finally, it will be exposed some weak relations between the coefficients of the DQE associated with the SM2 non-biquadratic equations; and these relations can also be verified in Examples 10–13.

Corollary 8. *If the GQE is an SM2 equation that corresponds to the Ferrari Case, then the following relations between the coefficients of the corresponding DQE hold:*

(i) $p^2 > -12r$;
(ii) $-2p\left(p^2 - 4r\right) \neq 9q^2 > 32pr$;
(iii) $-2p\left(p^2 - 36r\right) \neq 27q^2$.

Proof.

(i) Note that (ii) of Theorem 7 guarantees $\left(p^2 + 12r\right)^3 > 0$, thus $p^2 > -12r$.
(ii) If (ii) of Theorem 7 implies $p^2 + 12r > 0$, then Equation (40) and Remark 4 guarantee $9q^2 - 32pr > 0$ for all the SM2 non-biquadratic equations, thus $9q^2 > 32pr$; in addition, Equation (41) implies $2p\left(p^2 - 4r\right) + 9q^2 \neq 0$, thus $-2p\left(p^2 - 4r\right) \neq 9q^2$.
(iii) Note that (ii) of Theorem 7 also guarantees $\left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2 > 0$, so $p\left(p^2 - 36r\right) + \frac{27}{2}q^2 \neq 0$, thus $-2p\left(p^2 - 36r\right) \neq 27q^2$. □

5. Discussion and Results

5.1. Discussion on the Resolvent Cubic

In historical terms, most of the solutions given by the Classical Algebra to the GQE treat this equation as a two-in-one problem, giving one solution for the Biquadratic Case based on the Quadratic Formula, and giving other different solutions for the Ferrari Case based on the resolution of the Resolvent Cubic [1] (pp. 23–24); therefore, the existence of the Resolvent Cubic tends to be considered only for the Ferrari Case, in which all the roots of the Resolvent Cubic are non-zero.

Although Theorem 4 is not completely new, its novelty lies in featuring only one general formula for all cases of fourth-degree polynomial equations, although in practical

terms Equations (5) and (6) are usually easier to apply to solve biquadratic equations; nevertheless, in programing terms this theorem offers an only one general algorithm to solve the GQE, based on the definitions of set S and parameter α_s given by Equations (13) and (14), respectively.

In addition, although the existence and resolution of the Resolvent Cubic is usually disregarded in the Biquadratic Case, Theorem 4 shows how this third-degree equation can also be considered in this case; thus, one of the main consequences of this theorem is that it shows how Ferrari's Method or any known method that requires the roots of a Resolvent Cubic to solve only non-biquadratic quartic equations can also be generalized in a similar manner in order to solve all the quartic equations.

As it is exposed in [1] (p. 24), other important considerations on the Resolvent Cubic are related to its appearance, which depends on how it is obtained; for example, Theorem 4 is a generalized variation of the methods originally given by Ferrari and Descartes that are mainly used to solve non-biquadratic quartic equations, and all of these methods are based on the idea of expressing the quartic equations in terms of a product of two quadratic polynomials, so this inevitably implies the existence of the Resolvent Cubic.

Now consider the following polynomial functions:

- $R_C(s) := s^3 + 2ps^2 + (p^2 - 4r)s - q^2$
- $R_C^*(t) = t^3 - pt^2 - 4rt + (4pr - q^2)$
- $F_1(t) := 8t^3 + 8pt^2 + (2p^2 - 8r)t - q^2$
- $F_2(t) := 8t^3 - 4pt^2 - 8rt + (4pr - q^2)$
- $D(t) := t^6 + 2pt^4 + (p^2 - 4r)t^2 - q^2$
- $E(t) := t^3 + \frac{p}{2}t^2 + \left(\frac{p^2}{16} - \frac{r}{4}\right)t - \frac{q^2}{64}$

Then, it is clear that the Resolvent Cubic used in this paper can be briefly expressed as follows:

$$R_C(s) = 0 \tag{49}$$

However, if t had been cleared in Equation (17) and used in the proof of case 3 of Theorem 4 instead of s, then the Resolvent Cubic would have the form $R_C^*(t) = 0$; even more, Equation (17) also implies the relation $R_C(s) = R_C(t - p) = R_C^*(t)$.

Moreover, the cubic equations $F_1(t) = 0$ and $F_2(t) = 0$ are two alternative forms of the Resolvent Cubic given by the original Ferrari Method, according to [1] (pp. 23–24); now note that $R_C(2t) = F_1(t)$, so this also implies $R_C(2t - p) = F_1(t - p/2) = F_2(t)$; additionally, the Descartes Method gives the equation $D(t) = 0$ exposed in [1] (p. 66), which is also considered as a Resolvent Cubic although this one is a sixth-degree polynomial equation, because it can be solved as a third-degree equation due to the relation $R_C(t^2) = D(t)$; finally, the equation $E(t) = 0$ is the Resolvent Cubic given by the *Euler's Method* exposed in [8], now note that $R_C(4t)/64 = E(t)$.

So, there are three main reasons why Equation (49) should be considered as the "standard form" of the Resolvent Cubic for any quartic equation:

(i) All the known forms of the Resolvent Cubic can be expressed in terms of R_C.
(ii) Most of the forms of the Resolvent Cubic can be expressed as $a_2R_C(a_1t + a_0) = 0$, for some a_2, a_1, $a_0 \in \mathbb{R}$; so Remark 1, Lemma 1 and all the results presented here that depend on this lemma also hold for these forms, whenever $a_0 = 0$ (for example $F_1(t) = 0$ and $E(t) = 0$); otherwise, the hypotheses of all these results must be laid out again, according to the inevitable changes provoked by $a_0 \neq 0$, so this implies other uneasy complications.
(iii) Equation (49) is a monic polynomial equation that is also the most simplified and practical form of the Resolvent Cubic to study and explore some other relevant relations between the roots of any quartic equation and the roots of its Resolvent Cubic, which will be exposed in a subsequent article.

Likewise, all the forms of the Resolvent Cubic that can be expressed as $a_2R_C(a_1t + a_0) = 0$ with $a_0, a_1, a_2 \in \mathbb{R} - \{0\}$ should be considered as "translated forms" of the Resolvent Cubic

(for example $R_C^*(t) = 0$ and $F_2(t) = 0$), since Remark 1 and Lemma 1 do not necessarily hold for these forms of the Resolvent Cubic.

5.2. The Resolvent Cubic and the Quartic Equations with Multiple Roots

As it is stated in [7], it is well-known that any quartic equation has multiple roots if, and only if, its Resolvent Cubic has multiple roots, so the results presented here allow to appreciate some precisions about this fact; firstly, note that Theorems 5 and 6 guarantee that any non-biquadratic quartic equation and its Resolvent Cubic always have the same number of multiple roots; secondly, Theorems 1, 2 and 3 and Propositions 1 and 2 guarantee the following facts about the Resolvent Cubic for biquadratic equations with multiple roots:

- The Resolvent Cubic has the following form:

$$s^3 = 0\,, \tag{50}$$

Whenever the GQE is an M4 equation, so this is the only case where $s_1 = s_2 = s_3 = \alpha_s = 0$ and $S = \varnothing$.

- The Resolvent Cubic has the following form:

$$s^3 + 2ps^2 = (s + 2p)s^2 = 0\,, \tag{51}$$

Whenever the GQE is a DM2 equation, so its roots are $s_1 = -2p \neq 0$ and $s_2 = s_3 = 0$; hence, $\alpha_s = 0$ and $S = \{s_1\} \neq \varnothing$.

- The Resolvent Cubic has the following form:

$$s^3 + 2ps^2 + p^2 s = s(s + p)^2 = 0\,. \tag{52}$$

Whenever the GQE is an SM2 biquadratic equation, so its roots are $s_1 = 0$ and $s_2 = s_3 = -p \neq 0$; hence, $\alpha_s = 0$ and $S = \{s_2, s_3\} \neq \varnothing$.

Therefore, the GQE is an M4 or a DM2 equation if, and only if, zero is a multiple root of its Resolvent Cubic; complementarily, the SM2 biquadratic equations are the only ones of this kind of equations in which the multiple roots of the Resolvent Cubic are non-zero. Furthermore, although (i) of Lemma 1 guarantees that the biquadratic equations are the only quartic equations where zero is a root of the Resolvent Cubic, Equations (13) and (14) guarantee that Equation (15) never goes undefined.

5.3. How Does the Complex Arithmetic Become Unnecessary?

Moreover, consider the following facts that hold for quartic equations with real coefficients:

- The general formulae given by Equations (5) and (6) solve biquadratic equations without dealing with square roots of non-real complex numbers, whenever $p^2 \geq 4r$;
- Equation (15) solves biquadratic equations without dealing with square roots of non-real complex numbers, whenever $r \geq 0$;
- Remark 1 guarantees that all the non-biquadratic quartic equations can be solved without dealing with square roots of non-real complex numbers.

Hence, all these facts imply that any quartic equation with real coefficients can be solved avoiding the application of complex arithmetic operations, even though the quartic equation in question has non-real roots.

5.4. About the Nature of the Roots and the Relevance of the Standard Form of the Resolvent Cubic

Now consider that there are always only two possibilities for the nature of the roots of the Resolvent Cubic:

1. $\{s_1, s_2, s_3\} \subset \mathbb{R}$;
2. $s_1 \in \mathbb{R}$ and $\{s_2, s_3\} \subset \mathbb{C} - \mathbb{R}$, such that $s_2 = \overline{s_3}$.

So, if the GQE and its Resolvent Cubic have multiple roots, then the properties of complex numbers exposed in [2] imply that possibility 2 never happens because there cannot be s_2, $s_3 \in \mathbb{C} - \mathbb{R}$, such that $s_2 = s_3 = \overline{s_3}$; therefore, if the GQE has multiple roots, then the three roots of the Resolvent Cubic must always be real; additionally, if Δ_{GQE}, Δ_{DQE} and Δ are the respective discriminants of the GQE, of the DQE and of the Resolvent Cubic; then, according to [1] (p. 103), $\Delta_{GQE} = \Delta_{DQE} = \Delta \geq 0$ when the four roots of the GQE are real, and $\Delta_{GQE} = \Delta_{DQE} = \Delta = 0$ whenever the GQE has multiple roots (see also Appendix A).

However, Theorems 1, 5 and 6, Corollaries 1, 2 and 3 and Equations (49)–(52) allow going further because all together imply that any quartic equation with multiple roots has four real roots if, and only if, the standard form of the Resolvent Cubic has three non-negative real roots; so, it also holds for the non-translated forms of the Resolvent Cubic, and revealing this fact is a perfect example of the relevance and usefulness of the standard form of the Resolvent Cubic.

5.5. When Are the Resolvent Cubic and Its Roots Actually Necessary?

Likewise, the proofs of all the formulae exposed here to solve quartic equations with multiple roots show how the Tartaglia–Cardano Formulae or any other method for solving third-degree polynomial equations are actually unneeded to solve the corresponding Resolvent Cubic, in spite of the existence and the nature of this one, because solving this kind of quartic equations implies to solve only second-degree equations at most (see also Appendix B).

Furthermore, Corollaries 3, 6 and 7 show how the Resolvent Cubic becomes irrelevant to obtaining all the roots of M3 and SM2 non-biquadratic equations; although these ones correspond to the Ferrari Case. In fact, the only kind of quartic equations where the roots of the Resolvent Cubic are really indispensable, is the non-biquadratic equations without multiple roots; and this fact is also consistent with what is stated in Appendix B.

5.6. Criteria for Classifying the Quartic Equations with Multiple Roots

Finally, here are the following criteria, based on the coefficients of the DQE and the results exposed in this paper that can be very useful in programing terms to solve quartic equations:

(i) If $p^2 < -12r$ or $0 \leq \left(p^2 + 12r\right)^3 \neq \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2$, then it is an equation without multiple roots (Lemma 2).

(ii) If $p = q = r = 0$, then it is an M4 equation (Theorem 1).

(iii) If $p^2 = -12r > 0$ and $27q^2 = -8p^3 > 0$, then it is an M3 equation (Corollary 3 and Theorem 7).

(iv) If $p^2 = 4r > 0 = q$, then it is a DM2 equation (Theorem 2). In addition:
- If $p > 0$, then none of its four roots are real numbers (Corollary 1);
- If $p < 0$, then its four roots are real numbers (Corollary 1).

(v) If $p \neq q = r = 0$ or $q \neq 0 < \left(p^2 + 12r\right)^3 = \left[p\left(p^2 - 36r\right) + \frac{27}{2}q^2\right]^2$, then it is an SM2 equation (Theorems 3 and 7); in addition:
- If $p > 0 = q = r$ or if $p \geq 0 \neq q$ with $r > 0$ or if $p < 0 \neq q$ with $p^2 \leq 4r$, then only its non-multiple roots are real numbers (Corollaries 2 and 5 and Theorems 8 and 9);
- If $p < 0 = q = r$ or if $p < 0 \neq q$ with $p^2 > 4r$, then its four roots are real numbers (Corollaries 2 and 5 and Theorems 8 and 9).

6. Conclusions

Although the analytical solutions of third and fourth-degree equations have been known for half a millennium, this paper features some results that are generally ignored or overlooked by most of the texts dedicated to the theory of algebraic equations (Lemma 2, Remarks 1 and 3, Theorems 1, 2, 3, 5, 6, 7, 8, 9 and 10, and Corollaries 1, 2, 3, 4, 5, 6 and 7), so these results and the analyses presented here can help to facilitate the study and comprehension of this kind of equations and its analytical solutions, which are not as impractical as many people around the world tend to think.

So, this paper tries to offer a new approach of this kind of older analytical solutions by cases, with the main purpose of reassess them and made them "more friendly" in practical terms; because these ones are often overlooked or even looked down on, favoring instead the solutions given by the classical numerical methods that are usually applied to solve different kinds of scientific and practical problems that involve this kind of algebraic equations.

In addition, this work also might influence new ways of posing these old problems in educational terms, especially in a time like this when technology allows having a computing power never seen before; which can help to efficiently overcome all the difficulties that these old analytical solutions presented in previous centuries before the beginning of the computational era, providing more precise and efficient solutions to the problems mentioned in the previous paragraph; simply because these analytical solutions do not work approximating the solution by iterating, but they go straight to the precise solution right away.

In this sense and as mentioned in the introduction of this paper, this one is the first part of a complete research dedicated to fourth-degree polynomial equations with real coefficients that was focused on the following three issues:

1. The theoretical analysis of all quartic equations with real coefficients and multiple roots.
2. A new approach to the analytical solution of the GQE.
3. Some theoretical questions about the quartic discriminant.

Meanwhile, the second part of this research will be focused on the following three issues:

1. The analysis of the quartic equations without multiple roots that correspond to the Ferrari Case, which are the only kind of quartic equations that were not analyzed here.
2. A deeper analysis of the standard form of the Resolvent Cubic.
3. The design of a computer program based on those analyzes and the application of the formulae developed here that solve all the cases of quartic equations with multiple roots, in order to present an effective and general computational solution for the GQE without numerical methods and avoiding the application of complex arithmetic operations.

Author Contributions: Main ideas, proofs, discussion, results, conclusions, appendices and English—translation, M.C.-P.; Section 2, discussion, results, conclusions and Appendix A, M.A.M.-C.; examples, discussion, results and conclusions, A.T.-M.; examples, references and writing—review, D.M.-C.; writing—review and English—translation, T.A.-R. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding. The APC was funded by the TecNM–TESOEM.

Acknowledgments: To the TecNM–TESOEM and the Instituto Politécnico Nacional for all the support provided for the preparation of this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

According to the theory of polynomial equations, the discriminant of a quartic equation is defined as follows [1] (pp. 101–102):

$$\Delta_4 := [(u_1 - u_2)(u_1 - u_3)(u_1 - u_4)(u_2 - u_3)(u_2 - u_4)(u_3 - u_4)]^2 \,, \tag{A1}$$

where u_1, u_2, u_3 and u_4 are the four roots of this equation; then, note that Equation (1) guarantees $x_i - x_j = y_i - y_j \;\forall\; i, j \in \{1, 2, 3, 4\}$; hence, it is obvious that $\Delta_{GQE} = \Delta_{DGE}$.

Likewise, if y_1, y_2, y_3 and y_4 are the four roots of the DQE, then Viète Theorem, as stated in [4,5], implies the following relations between the roots and the coefficients of the DQE:

$$\begin{aligned} y_1 + y_2 + y_3 + y_4 = 0\,, \\ y_1y_2 + y_1y_3 + y_1y_4 + y_2y_3 + y_2y_4 + y_3y_4 = p\,, \\ y_1y_2y_3 + y_1y_2y_4 + y_1y_3y_4 + y_2y_3y_4 = -q\,, \\ y_1y_2y_3y_4 = r\,. \end{aligned} \tag{A2}$$

Since $y_4 = -y_1 - y_2 - y_3$, Equations (A1) and (A2) imply these other four relations:

- $p = -y_1^2 - y_2^2 - y_3^2 - y_1y_2 - y_1y_3 - y_2y_3$,
- $q = y_1^2y_2 + y_1^2y_3 + y_2^2y_3 + y_1y_2^2 + y_1y_3^2 + y_2y_3^2 + 2y_1y_2y_3$,
- $r = -y_1^2y_2y_3 - y_1y_2^2y_3 - y_1y_2y_3^2$,
- $\Delta_{DQE} = [(y_1 - y_2)(y_1 - y_3)(2y_1 + y_2 + y_3)(y_2 - y_3)(y_1 + 2y_2 + y_3)(y_1 + y_2 + 2y_3)]^2$;

Furthermore, all these relations imply the following equality after many tiresome algebraic calculations:

$$\frac{4\left(p^2 + 12r\right)^3 - \left[2p(p - 36r) + 27q^2\right]^2}{27} = \Delta_{DQE}\,. \tag{A3}$$

Therefore, Equations (37) and (A3) imply $\Delta_{DQE} = \Delta$; hence $\Delta_{GQE} = \Delta$ as well; so, this is another way to prove that the discriminant of any quartic equation always coincides with the discriminant of its Resolvent Cubic, and this is also the reason of the name given to Lemma 2.

Moreover, Equation (37) shows a very simplified form of Δ_{GQE} that depends only on the main three coefficients of the corresponding DQE; but if Equations (2)–(4) are applied to Equation (37), then the too much longer and explicit known form of Δ_{GQE} that depends on the five coefficients of the GQE is finally obtained:

$$\begin{aligned} \Delta_{GQE} = 256a^3e^3 - 192a^2bde^2 - 128a^2c^2e^2 + 144a^2cd^2e - 27a^2d^4 + \\ 144ab^2ce^2 - 6ab^2d^2e - 80abc^2de + 18abcd^3 + 16ac^4e - 4ac^3d - \\ 27b^4e^2 + 18b^3cde - 4b^3d^3 - 4b^2c^3e + b^2c^2d^2. \end{aligned} \tag{A4}$$

Appendix B

Since Equation (A4) allows to know immediately if the GQE has multiple roots, now it is introduced some general formulae to obtain all the multiple roots of any quartic equation with multiple roots, but without the help of the Resolvent Cubic.

Lemma A1. *If the DQE has multiple roots, then these ones are also roots of the following polynomial equation:*

$$2py^2 + 3qy + 4r = 0. \tag{A5}$$

Proof. If zero is a multiple root of the DQE, then $r = 0$, thus zero is also a root of Equation (A5). On the other hand, if $r \neq 0$ and the DQE has multiple roots, then there exist some t, $u \in \mathbb{C} - \{0\}$, such that the DQE can be expressed as follows:

$$(y - t)^2\left(y^2 + 2ty + u\right) = y^4 + \left(u - 3t^2\right)y^2 + 2\left(t^3 - tu\right)y + t^2u = 0; \tag{A6}$$

so, Equation (A6) implies the following non-linear system of equations:

$$3t^2 - u = -p; \tag{A7}$$

$$2\left(t^3 - tu\right) = q; \tag{A8}$$

$$t^2 u = r. \tag{A9}$$

Then, Equations (A7) and (A9) imply $3t^2 - u = 3t^2 - r/t^2 = -p$, thus:

$$3t^4 + pt^2 - r = 0; \tag{A10}$$

Simultaneously, Equations (A8) and (A9) imply $2(t^3 - tu) = 2\left[t^3 - t\left(r/t^2\right)\right] = 2t^3 - 2r/t = q$, thus:

$$2t^4 - qt - 2r = 0. \tag{A11}$$

Additionally, Equations (A10) and (A11) imply the relation $t^4 = \left(r - pt^2\right)/3 = (qt + 2r)/2$; then $2\left(r - pt^2\right) = 3(qt + 2r)$, so $2pt^2 + 3qt + 4r = 0$; therefore, t is a root of Equation (A5) and, according to Equation (A6), t is also a multiple root of the DQE. □

Theorem A1. *(General Formulae for the Multiple Roots of Quartic Equations) If the GQE has multiple roots, then all of them can be obtained by the following general formulae:*

(i) $x_M = \frac{bc-6ad \pm \sqrt{(bc-6ad)^2 - (8ac-3b^2)(16ae-bd)}}{8ac-3b^2}$, *when* $8ac \neq 3b^2$.
(ii) $x_M = \frac{16ae-bd}{2(bc-6ad)}$, *when* $8ac = 3b^2$ *and* $6ad \neq bc$.
(iii) $x_M = -\frac{b}{4a}$, *when* $8ac = 3b^2$ *and* $6ad = bc$.

Proof. First of all, note that Equations (1)–(4) and (A5) imply $2py^2 + 3qy + 4r = 2\left(\frac{8ac-3b^2}{8a^2}\right)\left(x + \frac{b}{4a}\right)^2 + 3\left(\frac{b^3-4abc+8a^2d}{8a^3}\right)\left(x + \frac{b}{4a}\right) + 4\left(\frac{16ab^2c-64a^2bd-3b^4+256a^3e}{256a^4}\right) = \left(\frac{8ac-3b^2}{4a^2}\right)x^2 + \left(\frac{6ad-bc}{2a^2}\right)x + \frac{16ae-bd}{4a^2} = 0$; thus, Equation (A5) is equivalent to the following equation:

$$\left(8ac - 3b^2\right)x^2 - 2(bc - 6ad)x + (16ae - bd) = 0. \tag{A12}$$

Additionally, Equation (1) and Lemma A1 guarantee that if the GQE has multiple roots, then these ones are also roots of Equation (A12).

(i) Now note that Equation (A12) is a quadratic equation whenever $8ac \neq 3b^2$, so in this case, the Quadratic Formula implies that the two roots of Equation (A12) are given by $x_M = \frac{-[-2(bc-6ad)] \pm \sqrt{[-2(bc-6ad)]^2 - 4(8ac-3b^2)(16ae-bd)}}{2(8ac-3b^2)} = \frac{bc-6ad \pm \sqrt{(bc-6ad)^2-(8ac-3b^2)(16ae-bd)}}{8ac-3b^2}$.
(i) If $8ac = 3b^2$ and $6ad \neq bc$, then Equation (A12) is reduced to a linear equation whose only root is $x_M = \frac{16ae-bd}{2(bc-6ad)}$.
(i) If $8ac = 3b^2$ and $6ad = bc$, then Equation (A12) holds whenever $16ae = bd$ as well; thus $c = 3b^2/8a$, $d = bc/6a = b\left(3b^2/8a\right)/6a = b^3/16a^2$ and $e = bd/16a = b\left(b^3/16a^2\right)/16a = b^4/256a^3$. Therefore, the GQE can be rewritten as follows: $ax^4 + bx^3 + \frac{3b^2}{8a}x^2 + \frac{b^3}{16a^2}x + \frac{b^4}{256a^3} = a\left(x + \frac{b}{4a}\right)^4 = 0$; so, this case is equivalent to Theorem 1. □

Corollary A1. If the DQE has multiple roots, then all of them are given as follows:

(i) $y_M = \frac{-3q \pm \sqrt{9q^2-32pr}}{4p}$, *when* $p \neq 0$;
(ii) $y_M = -\frac{4r}{3q}$, *when* $p = 0$ *and* $q \neq 0$;
(iii) $y_M = 0$, *when* $p = 0$ *and* $q = 0$.

Proof. This is the particular case of Theorem A1 when $a = 1$, $b = 0$, $c = p$, $d = q$ and $e = r$; and these formulae are also all the possible solutions of Equation (A5). □

Remark A1. *It is clear that the formulae of Theorem A1 and Corollary A1 can only be used to obtain the multiple roots of a quartic equation with multiple roots. So, in order to obtain its other roots, it will be necessary to apply factorization and the Quadratic Formula.*

Remark A2. *Note that (ii) and (iii) of Theorem A1 and (ii) and (iii) of Corollary A1 always give the multiple roots of the quartic equation in question. On the other hand, if (i) of Theorem A1 and (i) of Corollary A1 are applied to DM2 equations, then the two results are different among them, both of them can be real or non-real because Theorem 2 implies* $9q^2 - 32pr = -8p^3 \neq 0$*, and both results are the multiple roots of the equation in question; in fact, this is a consequence of the following relation that holds only for this case:* $ax^4 + bx^3 + cx^2 + dx + e = k\left[(8ac - 3b^2)x^2 - 2(bc - 6ad)x + (16ae - bd)\right]^2$*, for some* $k \in \mathbb{R} - \{0\}$*; additionally, if (i) of Theorem A1 and (i) of Corollary A1 are applied to M3 and SM2 biquadratic equations, then the two results obtained by these formulae are real and equal to each other because in these cases Theorem 3 and Corollary 3 imply* $9q^2 - 32pr = 0$*, so this guarantees that Equations (A5) and (A12) are quadratic equations with multiple roots; and finally, if (i) of Theorem A1 and (i) of Corollary A1 are applied to SM2 non-biquadratic equations, then (ii) of Corollary 8 implies* $9q^2 - 32pr > 0$*, thus the two results obtained by these formulae are real and different among them; however, this is the only case in which only one of these results is a multiple root of the quartic equation in question, whereas the other result is not even a root of this quartic equation.*

Finally, the effectivity of (i) of Theorem A1 can be verified in Examples 2–5, 9 and 11–13; the effectivity of (ii) of Theorem A1 can be verified in Example 10, the effectivity of (iii) of Theorem A1 can be verified in Example 1, and the effectivity of Corollary 6 can be verified in Example 8.

References

1. Tignol, J.P. *Galois' Theory of Algebraic Equations*, 2nd ed.; World Scientific Publishing Co. Pte. Ltd.: Hackensack, NJ, USA; London, UK, 2016.
2. Marsden, J.E.; Hoffman, M.J. *Análisis Básico de Variable Compleja/Basic Complex Analysis*, 2nd ed.; Editorial Trillas S.A. de C.V.: Mexico City, Mexico, 2012; pp. 29–30.
3. Uspensky, J.V. *Teoría de Ecuaciones/Theory of Equations*, 3rd ed.; Grupo Noriega Editores, Editorial Limusa S.A de C.V.: Mexico City, Mexico, 2005.
4. Weiss, M.J.; Dubisch, R. *Álgebra Superior/Higher Algebra for the Undergraduate*, 5th ed.; Limusa, S.A., Ed.; John Wiley: Mexico City, Mexico, 1980; pp. 108–110.
5. Viète Theorem—Encyclopedia of Mathematics. Available online: https://encyclopediaofmath.org/index.php?title=Vi%C3%A8te_theorem (accessed on 20 May 2022).
6. Stahl, S. *Introductory Modern Algebra: A Historical Approach*, 2nd ed.; John Wiley&Sons, Inc.: Hoboken, NJ, USA, 2013; p. 49.
7. Lin, V.Y. Algebraic Functions, Configuration Spaces, Teichmüller Spaces, and New Holomorphically Combinatorial Invariants. *Funct. Anal. Appl.* **2011**, *45*, 204–224.
8. Cauli, A. On the Resolution of Third and Fourth Degree Equations. *Int. J. Appl. Comput. Math* **2019**, *5*, 117. [CrossRef]